An introduction to the mathematics of gravitational waves

Áron Szabó

Nicolaus Copernicus University in Toruń

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The slogan of general relativity

Matter tells spacetime how to curve; spacetime tells matter how to move.

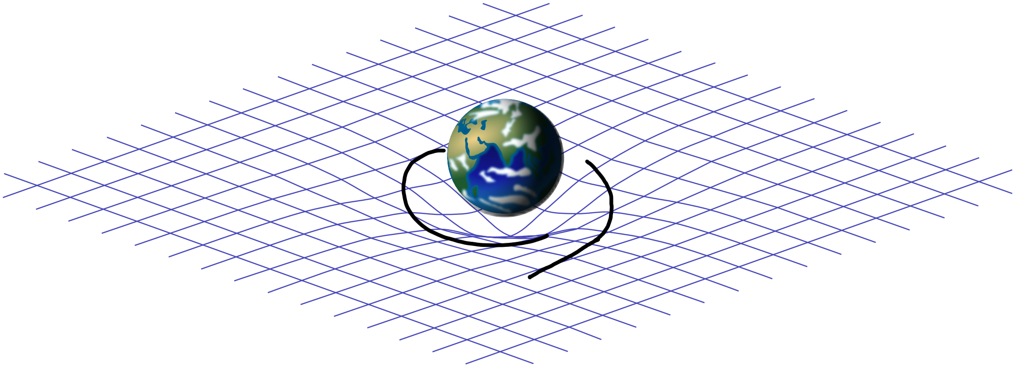


Image adapted from Wikipedia user Mysid

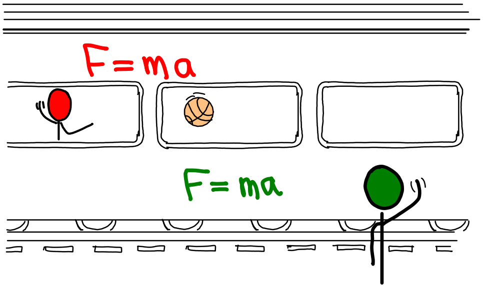
Principles of general relativity

Principle of local equivalence

In small, special relativity is a good approximation.

Principle of general covariance

The equations should transform covariantly under general transformations.



Units of measurement

We will work mostly in **geometrized units** where $c = 1$ and $G = 1$ (gravitational constant). This means we measure time, length and mass in the same units, say, seconds. If $[x]$ denotes the numerical value of a physical quantity in SI then in the geometrized system

$$1\,\mathrm{m} = \frac{1}{[c]}\,\mathrm{s} = \frac{1}{299792458}\,\mathrm{s}$$

$$1\,\mathrm{kg} = \frac{[G]}{[c^3]}\,\mathrm{s} = 2.477 \cdot 10^{-36}\,\mathrm{s}$$

Incidentally, modern SI (since 2019) is also based on the idea of fixing units of measurements via universal constants but there the natural constants have more complicated values for historical reasons.

A quick and dirty crash course in differential geometry

metric

geodesics — curvature — volume

normal coord — geodesic deviation — Ricci tensor

scalar

The metric

v^i

$\vec{v}, \vec{w}$ $\quad \boxed{\vec{v}} \cdot \vec{w} = (v^1\ v^2)\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = (v^1\ v^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} w^1 \\ w^2 \end{pmatrix}$

Covector $\quad v_i$

- Recall that everything about Euclidean geometry is encoded in the scalar product.
- The central object of general relativity is the spacetime metric $ds^2(x) = \sum_{i,j=1}^{4} g_{ij}(x) dx^i dx^j$
  - nondegenerate symmetric, bilinear form on tangent vectors
  - at each spacetime point, it can be written in some coordinates as $\begin{pmatrix} -\mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix}$ (however, generically this holds only at the given point, see later)
- note that the metric is not positive definite $\rightsquigarrow$ causal structure
- one can measure **time intervals** along timelike vectors, **lengths** of spacelike vectors and **angles** between two spacelike vectors

Example: special relativity

The spacetime of special relativity is the affine space $\mathbb{R}^4$ endowed with the metric given by the **constant** matrix

$$\eta_{mn}(x) \equiv \begin{pmatrix} \mathbf{-1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix}$$

in standard coordinates. We will see later that this spacetime is empty.

The kinematics of curves

Given a (parametrized) curve $x^l(s)$ in M, we call the quantities

$$v^l(s) := \dot{x}^l(s)$$

the **velocity** of the curve, and $\quad v := \dot{\gamma}$

$$a^l(s) := \ddot{x}^l(s) + \sum_{j,k=1}^{4} \Gamma^l{}_{jk}(x(s)) \cdot v^j(s) \cdot v^k(s)$$

the **acceleration** of the curve $\quad a_\gamma := \nabla_{\dot{\gamma}}\dot{\gamma}$

The coefficient functions are called the **Christoffel symbols** and they can be calculated as

$$\Gamma^l{}_{jk}(x) = \sum_{r=1}^{4} \frac{1}{2} g^{lr}(x) \cdot \left(\partial_k g_{rj}(x) + \partial_j g_{rk}(x) - \partial_r g_{jk}(x)\right) \qquad \nabla_{\partial_j}\partial_k = \Gamma^l{}_{jk}\partial_l$$

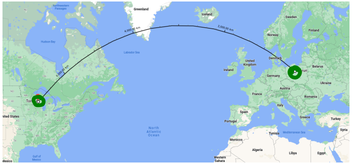
where g^{lr} is the inverse matrix of g_{ij}.

As straight as it gets – geodesics

We say a curve is a **geodesic** if its acceleration is always zero.

$$\ddot{x}^l(s) + \Gamma^l{}_{jk}(x(s)) \cdot \dot{x}^j(s) \cdot \dot{x}^k(s) = 0$$

This equation has a solution for a short while for any initial conditions.
Moreover, on a small enough scale there is a single geodesic between two points.



Spacetime tells matter how to move

Timelike geodesics describe **freely falling particles** (under the influence of gravitation only).

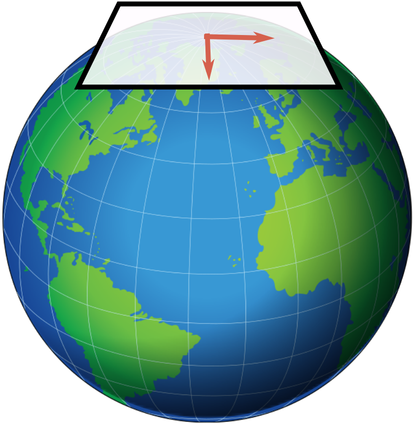
Image credit Google Maps and Openstreetmap

Intermezzo – Santa Claus and the normal coordinates

- Santa Claus invests in cheap drones
- they travel with constant velocity for 12 hours
- they need to be started together

Image adapted from Wikipedia

Intermezzo – Santa Claus and the normal coordinates



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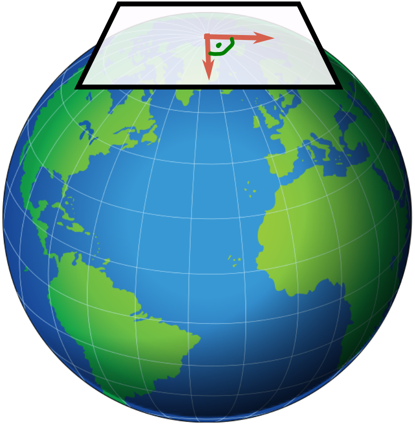


- Santa Claus invests in cheap drones
- they travel with constant velocity for 12 hours
- they need to be started together
- their initial velocity determines where they end up
- Santa may choose two orthogonal unit vectors to construct a coordinate system
- **normal coordinates** around the north pole

Image adapted from Wikipedia

The curvature tensor

$$\begin{pmatrix} -1 & & 0 & \\ 0 & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

- Normal coordinates in general with the same procedure
- Taylor expansion of the metric in these coordinates:

$$g_{ij}(x) = \eta_{ij} + 0 + R_{ijkl}x^k x^l + \mathcal{O}(|x|^3)$$

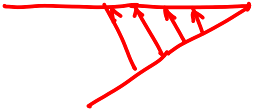
We call R_{ijkl} the **Riemann curvature tensor**.

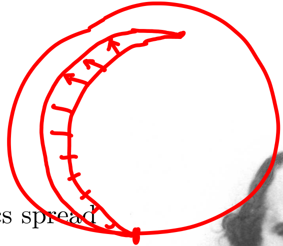
- **local equivalence principle**.
- The curvature tensor can be expressed in terms of the Christoffel symbols

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}$$

$$= R(g_{mn}, \partial_p g_{mn}, \partial_p \partial_q g_{mn})$$

Geodesic deviation

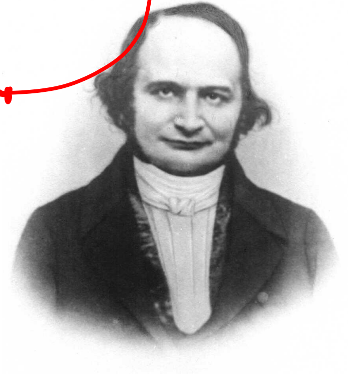




The Riemann curvature tensor governs how nearby geodesics spread out via this differential equation

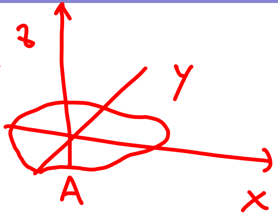
$$\frac{\nabla^2 \boxed{n^a(s)}}{ds^2} = R^a{}_{bcd}(u(s)) \cdot u^b(s) \cdot u^d(s) \cdot \boxed{n^c(s)},$$

where $u^a(s)$ is the geodesic around which we do our investigation.



Volumes and the Ricci curvature

Jacobi determinant



- Recall the formula for integration in polar coordinates:

$$\iint_A f(x,y)\,dx\,dy = \iint f(r,\phi)\,r\,dr\,d\phi.$$

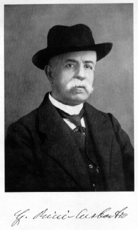
- On a spacetime, this generalizes to $\int_A f(x)\boxed{\sqrt{|\det g_{ij}|}}d^4x$, and

$$\mathrm{Vol}(A) = \int_A \sqrt{|\det g_{ij}|}\,d^4x.$$

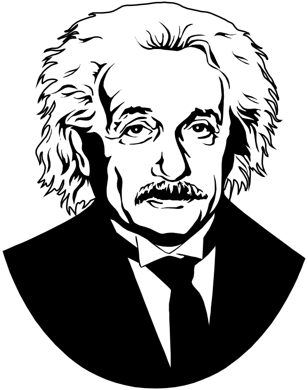
- In normal coordinates we have the expansion

$$\sqrt{|\det g_{ij}|} = 1 + 0 - \frac{1}{6}R^k{}_{ikj}x^i x^j + \mathcal{O}(|x|^3)$$

- We call $R_{ij} := R^k{}_{ikj}$ the **Ricci tensor** of the metric
- Moreover, its trace $S := R^i_i$ is called the **scalar curvature** of the metric.



Einstein's equations



Desiderata

Motivation: physics is full of **second-order differential equations**:

$$m\ddot{r} = \mathbf{F}(\dot{r}, r, t) \tag{Newton}$$

$$\begin{cases} \Delta\varphi + \frac{\partial}{\partial t}\operatorname{div}\mathbf{A} & = -\frac{\rho}{\varepsilon_0} \\ \Delta\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \operatorname{grad}\left(\operatorname{div}\mathbf{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t}\right) & = -\mu_0\mathbf{J} \end{cases} \tag{Maxwell}$$

Goal:

$$\boxed{\begin{gathered}\text{some second-order expression in } g = \text{matter content of the universe} \\ \Theta_{ab}(g_{mn}, \partial_p g_{mn}, \partial_p\partial_q g_{mn}) = \text{const}\cdot T_{ab}\end{gathered}}$$

Ideally, the left-hand side should be divergence free and quasilinear.
Moreover, by the principle of covariance, we require that the equation be generally covariant[1].

[1] That is, it should be covariant under all diffeomorphisms.

The Einstein equations are canonical 1

Theorem 1 (Lovelock, 1971)

In four dimensions, the only such Θ_{ab} *are of the form*

$$\Theta = \alpha \cdot G_{ab} + \Lambda \cdot g_{ab},$$

where $\alpha, \Lambda \in \mathbb{R}$ *are constants and*

$$G(g)_{ab} := R(g)_{ab} - \frac{1}{2} S(g) \cdot g_{ab}$$

is the so-called ***Einstein tensor****.*

The Einstein equations are canonical 2

Einstein equation!

$$\alpha(R_{ab} - \frac{1}{2}S \cdot g_{ab}) + \Lambda \cdot g_{ab} = \text{const} \cdot T_{ab}$$

- Based on experience in physics, we want to have a differential equation, thus $\alpha \neq 0$ and we can assume $\alpha = 1$ by rescaling.

The Einstein equations are canonical 2

Einstein equation!

$$R_{ab} - \frac{1}{2} S \cdot g_{ab} + \Lambda \cdot g_{ab} = \text{const} \cdot T_{ab}$$

- Based on experience in physics, we want to have a differential equation, thus $\alpha \neq 0$ and we can assume $\alpha = 1$ by rescaling.
- The constant turns out to be 8π in our system of units.

The Einstein equations are canonical 2

Einstein equation!

$$R_{ab} - \frac{1}{2} S \cdot g_{ab} + \Lambda \cdot g_{ab} = 8\pi T_{ab}.$$

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- The constant turns out to be 8π in our system of units.
- For the sake of simplicity, we will take $\Lambda = 0$ in this lecture.

The Einstein equations are canonical 2

Einstein equation!

$$\underbrace{R_{ab} - \frac{1}{2}S \cdot g_{ab}}_{G(g)} + \cancel{\Lambda \cdot g_{ab}} = 8\pi T_{ab}.$$

$$G(g) = 8\pi T(g)$$

- Based on experience in physics, we want to have a differential equation, thus $\alpha \neq 0$ and we can assume $\alpha = 1$ by rescaling.
- The constant turns out to be 8π in our system of units.
- For the sake of simplicity, we will take $\Lambda = 0$ in this lecture.
- We have seen that the Ricci tensor tells us how the volumes change, which plays nicely with the interpretation of curvature as gravity.

The Einstein equations are hard so solve

$$g_{ab}(x) \equiv \begin{pmatrix} -1 & & & \\ & 1 & 0 & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

In terms of the metric, the Einstein equation looks like this:[2]

$$\begin{aligned}
&\frac{1}{2}g^{ab}\underline{\partial_a\partial_m g_{bn}} + \frac{1}{2}g^{ab}\underline{\partial_a\partial_n g_{mb}} - \frac{1}{2}g^{ab}\underline{\partial_a\partial_b g_{mn}} - \frac{3}{2}g^{ab}\underline{\partial_m\partial_n g_{ab}} \\
&-\frac{1}{2}g^{bl}g^{ar}\partial_a g_{rl}\underline{\partial_m g_{bn}} - \frac{1}{2}g^{bl}g^{ar}\partial_a g_{rl}\underline{\partial_n g_{mb}} + \frac{1}{4}g^{bl}g^{ar}\partial_n g_{al}\underline{\partial_m g_{rb}} \\
&+\frac{1}{4|g|}g^{ab}\partial_b|g|\underline{\partial_n g_{ma}} - \frac{1}{4|g|}g^{ab}\partial_b|g|\underline{\partial_a g_{mn}} - \frac{1}{4|g|}g^{ab}\partial_b|g|\underline{\partial_m g_{an}} \\
&+\cancel{\Lambda g_{mn}} \qquad = 8\pi T_{mn}.
\end{aligned}$$

Finding solutions is hopeless except for **highly symmetric** situations (Minkowski, Schwarzschild, Robertson–Walker etc.) or with **extra conditions** (global hyperbolicity, Choquet-Bruhat etc.).

[2] The credit for the explicit formula goes to Ville Hirvonen from Profound Physics.

Linearization

This is where the linearized theory comes into play: suppose we have a "background" solution g_{ab} of the full Einstein equation

$$G(g) = 8\pi T(g)$$

and look for solutions of the form

$$g_{ab} + \epsilon h_{ab}$$

where h_{ab} is a symmetric tensor.

$$G(g + \epsilon h) = 8\pi T(g + \epsilon h)$$

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$$\cancel{G(g)} + \epsilon \cdot \boxed{\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} G(g + \epsilon h)} + O\left(\epsilon^2\right) = \cancel{8\pi T(g)} + \epsilon \cdot \boxed{8\pi \cdot \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} T(g + \epsilon h)} + O\left(\epsilon^2\right)$$

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$$G(g) + \epsilon \cdot \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} G(g + \epsilon h) + O\left(\epsilon^2\right) = 8\pi T(g) + \epsilon \cdot 8\pi \cdot \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} T(g + \epsilon h) + O\left(\epsilon^2\right)$$

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} G(g + \epsilon h) = 8\pi \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} T(g + \epsilon h)$$

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline t & -1 & & & \\ x & & 1 & & \\ y & & & 1 & \\ z & & & & 1 \end{array}$$

$f(t,x)$

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} G(g+\epsilon h) = 8\pi \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} T(g+\epsilon h)$$

$$-\frac{1}{k}\frac{\partial^2}{\partial t^2} f(t,x) + \frac{\partial^2}{\partial x^2} f(t,x) = 0$$

speed of the wave $\sqrt{k}$

Simplifying assumptions:

- the right-hand side is zero (that is, the perturbation may contribute only from the second order term).
- We linearize around the Minkowski metric, that is $g_{mn} = \eta_{mn}$ and $T(g) = 0$.

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} G(\eta+\epsilon h) = 0$$

After some tedious calculations, one arrives at

$$-\frac{\partial^2}{\partial t^2} h_{mn} + \frac{\partial^2}{\partial x^2} h_{mn} + \frac{\partial^2}{\partial y^2} h_{mn} + \frac{\partial^2}{\partial z^2} h_{mn}$$

$$\eta^{ab}\partial_a\partial_b h_{mn} \quad \boxed{\square h = 2\,\mathrm{div}(\nabla h) - \mathrm{Hess}(\mathrm{tr}\, h)} = \left(-\frac{\partial^2}{\partial t^2} + \Delta\right) h_{mn}$$

This is almost a wave equation for h_{ab}! But there are a few annoying terms.

Intermezzo – gauge freedom

In many physical theories there are extra degrees of freedom.

$$m\ddot{r} = \mathbf{F}(\dot{r}, r, t) = -\operatorname{grad} V(r) \qquad V \rightsquigarrow V + c \quad \text{gauging}$$

$$\begin{cases} \Delta\varphi + \frac{\partial}{\partial t}\operatorname{div}\mathbf{A} & = -\frac{\rho}{\varepsilon_0} \\ \Delta\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \operatorname{grad}\left(\operatorname{div}\mathbf{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t}\right) & = -\mu_0\mathbf{J} \end{cases} \qquad \begin{cases} \varphi \rightsquigarrow \varphi - \frac{\partial\lambda}{\partial t} \\ \mathbf{A} \rightsquigarrow \mathbf{A} + \operatorname{grad}\lambda \end{cases}$$

Using this freedom, we may be able to **impose conditions** that make calculations easier, e.g.

- that the mechanical potential is zero at a given point
- that the 4-divergence of the potential is zero $\operatorname{div}\mathbf{A} + \mu_0\varepsilon_0\frac{\partial\varphi}{\partial t} = 0$ (Lorenz gauge)

In general, if we want to achieve a certain condition, we need to solve a partial differential equation for the gauging terms, or at least argue why there is a solution.

Gauge fixing

The Einstein equation is generally covariant, therefore there is a huge amount of gauge freedom. On a linear level, this manifests itself like this:

Fact: vacuum gauge freedom around the Minkowski metric

If h_{ab} solves the linear vacuum Einstein equations around η, then so does $h_{ab} + \boxed{\partial_b X_a + \partial_a X_b}$ for any vector field X^a.

Such a change can be described as letting the metric flow under the vector field[3] $X^i \partial_i$, and it does not change the underlying physics due to general covariance. However, the calculations are greatly simplified with this extra term.

Fact (**partial** gauge fixing by Lorenz gauge)

By choosing the vector field appropriately, we can achieve that the 4-divergence of $h^{\mathrm{Lor}}_{ab} := h_{ab} + X_{a,b} + X_{b,a}$ is zero.

$h_{ab} + \partial_b X + \partial_a X_b$

[3] More precisely, the extra term is the coordinate expression of the Lie derivative $\mathcal{L}_X g$.

Vacuum and the wave equation

After some calculations one arrives at the following equation (here $\bar{h}_{ab}$ is the trace-reversed version of h_{ab}).

The linearized vacuum Einstein equations in Lorenz gauge

$$0 = \bar{h}^{;a}_{mn,a} = \eta^{ab}\bar{h}_{mn,ab} = \left(-\frac{\partial^2}{\partial t^2} + \Delta\right)\bar{h}_{ab}$$

Key result

Linear perturbations solve a wave equation and thus propagate with the speed of light in vacuum.

Plane waves

constant matrix of amplitudes

- Plane wave ansatz:

wave number vector

$$\bar{h}_{ab} = A_{ab} \cos(k_m x^m)$$

- This is a single Fourier mode
- Divergence-freeness implies $k^a A_{ab} = 0$.

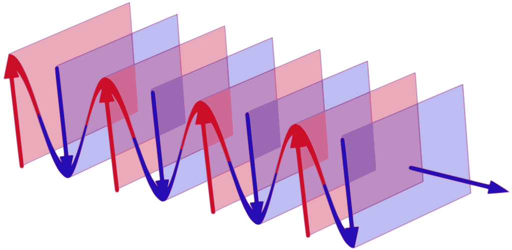

Image credit Wikipedia user Constant314

Finishing fixing the gauge

The Lorentz gauge is only a partial gauge, we may impose further conditions.

Further gauge fixing

By adding a further gauging term, we may further achieve that

- the trace is zero $\bar{h}_{ab}$
- $A_{0b} = 0$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & * & \\ 0 & & & \end{pmatrix}$$

Our plane wave has only two degrees of freedom

Next suppose we orient our spatial coordinate axes so that the wave is travelling in the positive z-direction, i.e.

$$k^t = \omega, \quad k^x = k^y = 0, \quad k^z = \omega$$

and

$$k_t = -\omega, \quad k_x = k_y = 0, \quad k_z = \omega$$

$$\begin{pmatrix} -\omega \\ 0 \\ 0 \\ \omega \end{pmatrix}$$

Then $A_{az} = 0$ for all a.
All in all, we obtain

$$\bar{h}_{mn} = \begin{pmatrix} A_{tt} & A_{tx} & A_{ty} & A_{tz} \\ A_{xt} & A_{xx} & A_{xy} & A_{xz} \\ A_{yt} & A_{yx} & A_{yy} & A_{yz} \\ A_{zt} & A_{zx} & A_{zy} & A_{zz} \end{pmatrix} \cos(\omega(t - z))$$

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and

$$k_t = -\omega, \quad k_x = k_y = 0, \quad k_z = \omega$$

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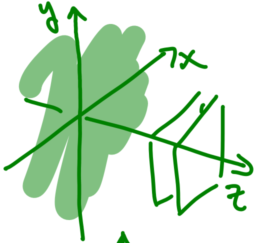
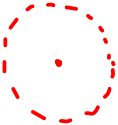
Effects of free particles: a ring of test particles

Consider a particle at rest at the origin with respect to a Lorentz frame and another particle which is initially $(x, y, z) = (\epsilon \cos\theta, \epsilon \sin\theta, 0)$ (i.e. on a circle perpendicular to the wave propagation direction).
Then one can show that the test particle stays at the origin, and for the deviation vector between the two particles ξ^a satisfies the equations



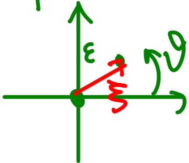
$$\begin{cases} \frac{\partial^2}{\partial t^2}\xi^x &= \frac{1}{2}\epsilon\cos\theta\frac{\partial^2}{\partial t^2}h_{xx} + \frac{1}{2}\epsilon\sin\theta\frac{\partial^2}{\partial t^2}h_{xy} \\ \frac{\partial^2}{\partial t^2}\xi^y &= \frac{1}{2}\epsilon\cos\theta\frac{\partial^2}{\partial t^2}h_{xy} - \frac{1}{2}\epsilon\sin\theta\frac{\partial^2}{\partial t^2}h_{xx}, \end{cases}$$

which has solution

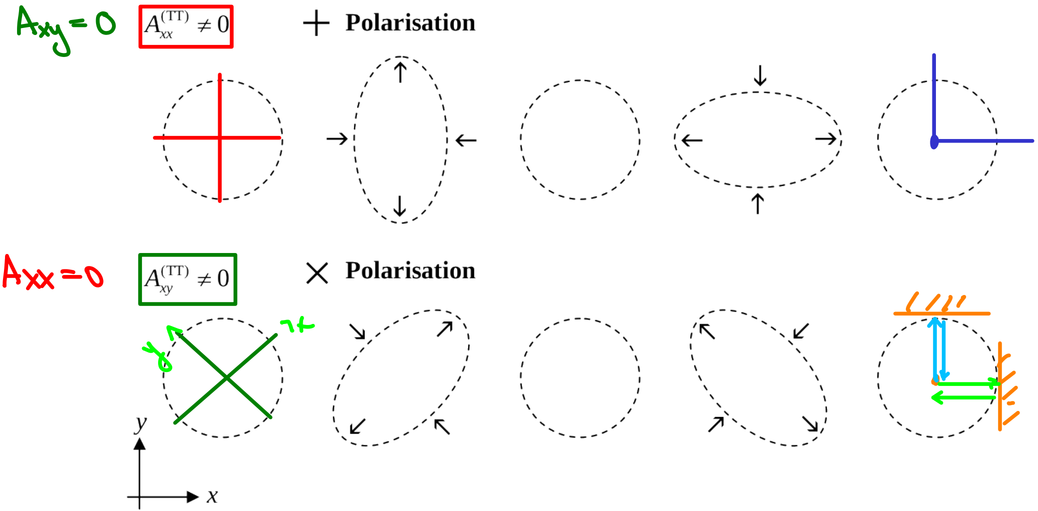


$$\xi^x = \epsilon\cos\theta + \frac{1}{2}\epsilon\cos\theta \boxed{A_{xx}} \cos(\omega t) + \frac{1}{2}\epsilon\sin\theta \boxed{A_{xy}} \cos(\omega t)$$

$$\xi^y = \epsilon\sin\theta + \frac{1}{2}\epsilon\cos\theta \boxed{A_{xy}} \cos(\omega t) - \frac{1}{2}\epsilon\sin\theta \boxed{A_{xx}} \cos(\omega t)$$



Polarisation states



How to detect gravitational waves?

We cannot measure the deviations with sticks since they are also subject to the same effect. However, a Michelson–Morley interferometer can measure the round-time in both directions.

$$\frac{\Delta L}{L} \approx 10^{-21}$$

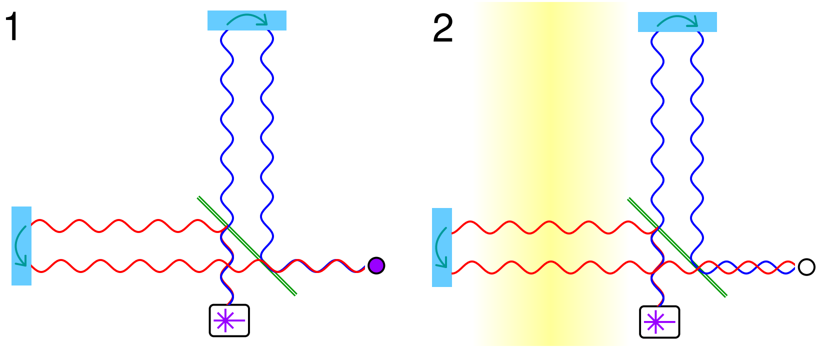


Image adapted from Wikipedia user Cmglee.

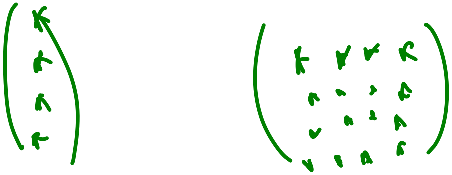
Summary

- We have seen how the Einstein equations follow from first principles.
- Since it is nonlinear, we linearized it.
- In an appropriately chosen gauge, the linearization is a wave equation.
- We solved this equation with a plane wave ansatz.
- These gravitational waves cause a wobble that propagate with the speed of light.
- This can be measured with interferometers.

But wait. . . there is more!

- What happens if we choose a different background solution?
- Do gravitational waves carry energy?
- What is the meaning of gravitational waves in the full theory?
- How do gravitational waves form?
- etc.

Suggested reading



calculation details about the plane wave

- Chapter 20 in R. D'Inverno, Introducing Einstein's relativity, Clarendon Press, 1998
- B. F. Schutz's lecture notes about Gravitational Waves at the 2011 Azores School on Observational Cosmology (online) ← he talks about detectors, too
- T. Matolcsi, Spacetime without reference frames, Minkowski Press, 2020 (online) ← a clean treatment of elementary spacetimes

$$\Box h_{ab} + R * h = 0$$