Numerical Approaches to the Relativistic Two-Body Problem: the Evolution Equations

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Overview

- Evolution equations
- Focus on issues arising in numerical implementations
- Current status of simulations for
  - black holes
  - neutron stars

3+1 Decompositions

Einstein's equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

for metric

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

can be split into

- constraint equations (like "div" equations in E&M)

$$R + K^2 - K_{ij}K^{ij} = -16\pi\rho \qquad \text{(Hamiltonian constraint)}$$

$$D_i(K^{ij} - \gamma^{ij}K) = 8\pi j^j \qquad \text{(Momentum constraint)}$$

- evolution equations (like "curl" equations)

$$\frac{d}{dt}\gamma_{ij} \equiv \left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)\gamma_{ij} = -2\alpha K_{ij}$$

$$\frac{d}{dt}K_{ij} = -D_iD_j\alpha + \alpha(R_{ij} - 2K_{ik}K^k{}_j + KK_{ij} - 8\pi M_{ij})$$

$\Longrightarrow$ "ADM" equations
[Arnowitt, Deser & Misner, 1962]

To construct initial data...
$\Longrightarrow$ ... solve constraint equations (Eric's talk)

To study evolution...
$\Longrightarrow$ ... solve evolution equations (this talk)

Evolution Calculations

- Take initial data γ_{ij} and K_{ij}
- Choose coordinate system α and β^i
- Evolve with evolution equations

$$\frac{d}{dt}\gamma_{ij} = -2\alpha K_{ij}$$

$$\frac{d}{dt}K_{ij} = -D_i D_j \alpha + \alpha(R_{ij} - 2K_{ik}K^k{}_j + KK_{ij} - 8\pi M_{ij})$$

where

$$R_{ij} = -\frac{1}{2}\gamma^{kl}(\gamma_{ij,kl} + \gamma_{kl,ij} - \gamma_{kj,il} - \gamma_{il,kj}) + \dots$$

$\Longrightarrow$ Can bring simple wave equation $(\partial_t^2 - \nabla^2)\phi = 0$ into very similar structure

$$\frac{d}{dt}\phi = -\psi$$

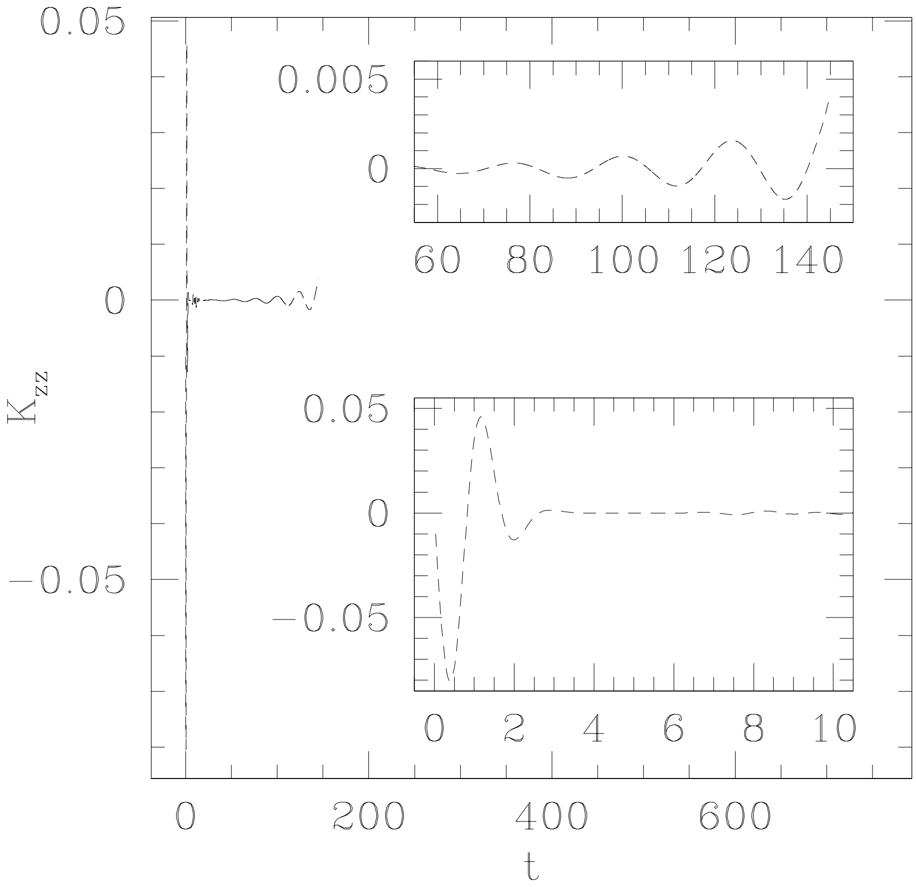
$$\frac{d}{dt}\psi = -\nabla^2\phi = -\gamma^{kl}\phi_{,kl}$$

$\Longrightarrow$ very simple to implement numerically

$\Longrightarrow$ Try!

Evolution of small amplitude waves

Use small amplitude gravitational waves ("Teukolsky waves") as initial data and evolve with ADM



Two problems with ADM equations

- principal part
- source terms

Principle Part - Comparison with E&M

Maxwell's equations:

- "constraint" equations

$$D_i E^i = 4\pi\rho_e$$

$$D_i E^i = 0$$

- "evolution" equations

$$\partial_t E_i = \epsilon_{ijk} D^j B^k - 4\pi J_i$$

$$\partial_t B_i = -\epsilon_{ijk} D^j B^k$$

Introduce vector potential A_i so that

$$B^i = \epsilon^{ijk} D_j A_k$$

Then Maxwell's equations in "ADM" form

$$D_i E^i = 4\pi\rho_e$$

$$\partial_t A_i = -E_i - D_i\Phi$$

$$\partial_t E_i = -D^j D_j A_i + D_i D^j A_j - 4\pi J_i$$

$\Longrightarrow$ Mixed second derivatives as in Ricci tensor R_{ij}

$\Longrightarrow$ Spoil hyperbolicity

How to get rid of offending term

Can eliminate "offending term" $D_i D^j A_j$ in

$$\partial_t A_i = -E_i - D_i \Phi$$

$$\partial_t E_i = -D^j D_j A_i + D_i D^j A_j - 4\pi J_i$$

in various different ways

- Gauge choice:
  E.g. Coulomb gauge $D^j A_j = 0$. Equivalent in GR: harmonic gauge. Usually undesirable, since would like to maintain gauge freedom

- Bring equations into hyperbolic form:
  Take time derivative of $\partial_t E_i$ equation, insert $\partial_t A_i$ equation, yields

  $$-\partial_t^2 E_i + D_j D^j E_i = 4\pi(\partial_t J_i + D_i \rho_e)$$

  $\Longrightarrow$ Wave equation for E_i

- Introduce auxiliary variable:
  Define $\Gamma \equiv D^j A_j$ to replace $D_i D^j A_j$ with $D_i \Gamma$ in $\partial_t E_i$ equation.

Rewrite Maxwell's equations

With $\Gamma \equiv D^j A_j$ get

$$\partial_t A_i = -E_i - D_i \Phi$$
$$\partial_t E_i = -D^j D_j A_i + D_i \Gamma - 4\pi J_i$$

and

$$\partial_t \Gamma = D^j \partial_t A_i = -D^j E_j - D^j D_j \Phi = -4\pi\rho - D^j D_j \Phi$$

How can these manipulations affect numerical stability??

$\Longrightarrow$ Constraint violations!

Compare propagation of constraint violations

$$\mathcal{C} \equiv D_i E^i - 4\pi\rho$$

- In "ADM" form:

$$\partial_t \mathcal{C} = 0$$

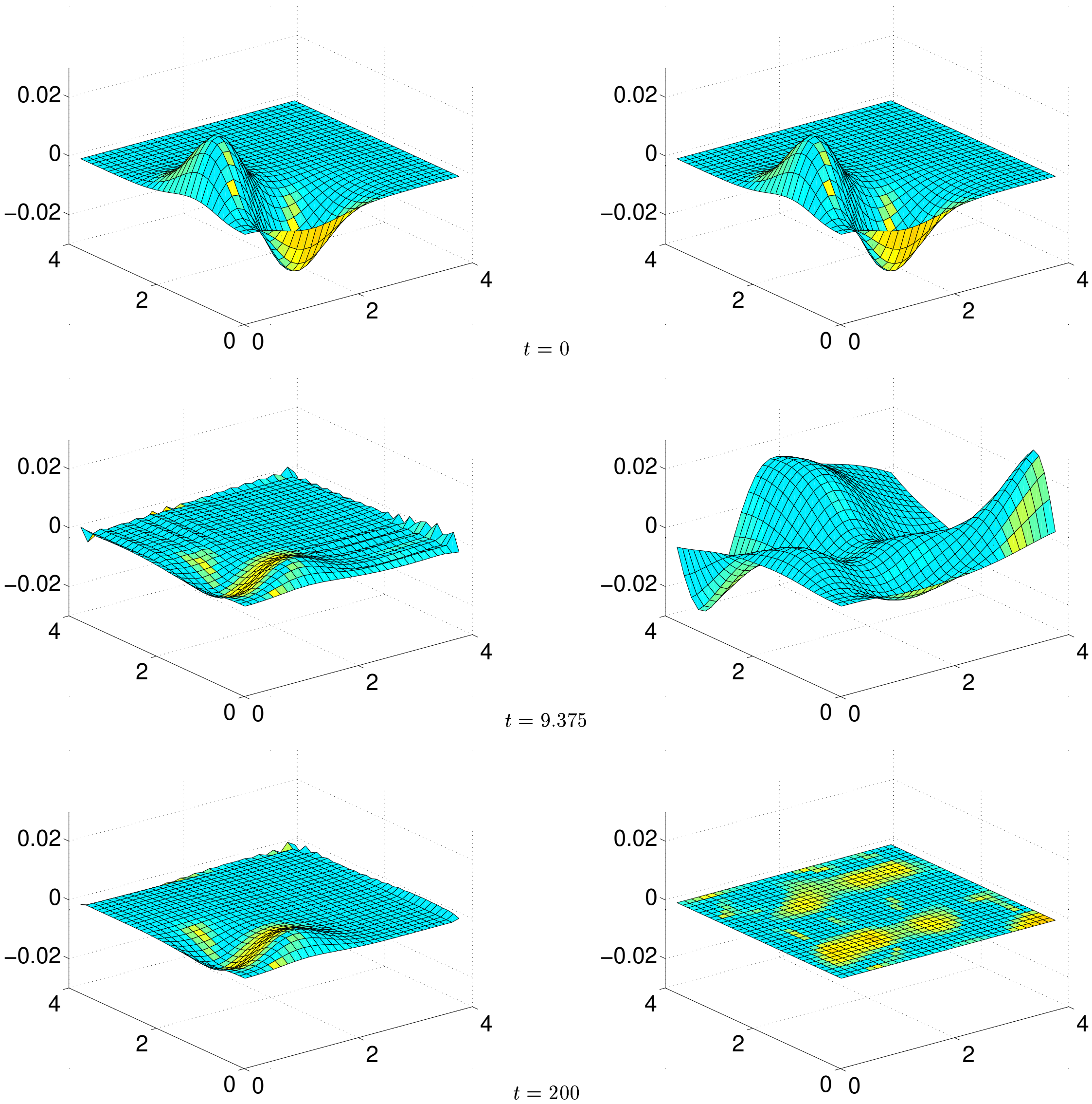
  $\Longrightarrow$ constraint violations don't "go away"
  $\Longrightarrow$ may cause instabilities in non-linear theories

- In rewritten form:

$$\partial_t^2 \mathcal{C} = D_i D^i \mathcal{C}$$

  $\Longrightarrow$ constraint violations satisfy wave equation
  $\Longrightarrow$ propagate off numerical grid

Numerical comparison



[Knapp, Walker & Baumgarte, 2002]

Rewrite ADM equations

Can rewrite ADM equations in completely analogous way

- use conformal transformation (so that $\det(\bar{\gamma}_{ij}) = 1$)
- introduce

$$\bar{\Gamma}^i \equiv \bar{\gamma}^{lm}\bar{\Gamma}^i_{lm} = -\bar{\gamma}^{ij}{}_{,j}$$

(compare with $\Gamma = D_i A^i$)

$\Longrightarrow$ new "constraint":

$$\mathcal{G}^i \equiv \bar{\Gamma}^i - \bar{\gamma}^{lm}\bar{\Gamma}^i_{lm} = 0$$

Then

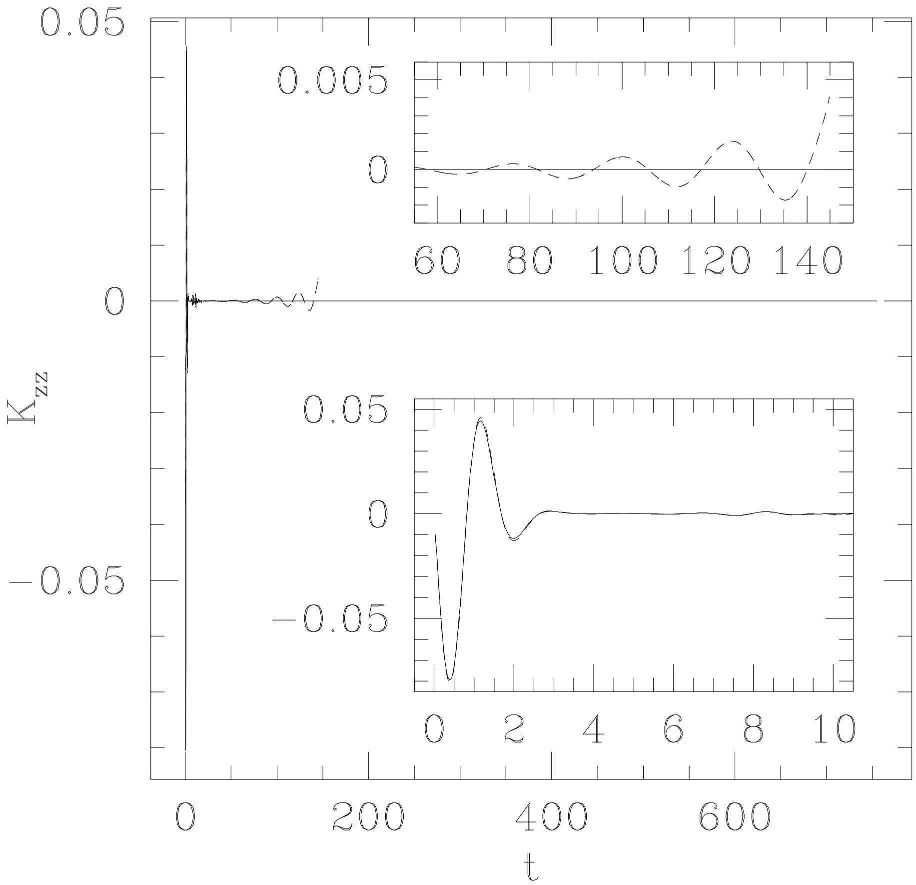
$$\bar{R}_{ij} = -\frac{1}{2}\bar{\gamma}^{lm}\bar{\gamma}_{ij,lm} + \bar{\gamma}_{k(i}\partial_{j)}\bar{\Gamma}^k + \ldots$$

and

$$\partial_t\bar{\Gamma}^i = -\partial_j(\partial_t\bar{\gamma}^{ij}) \sim \partial_j\bar{A}^{ij} + \ldots \sim 8\pi j^i + \ldots$$

$\Longrightarrow$ "BSSN"
[Shibata & Nakamura, 1995; Baumgarte & Shapiro, 1999]

Revisit evolution of small amplitude waves



[Baumgarte & Shapiro, 1999]

Improved numerical stability related to propagation of constraint violations
[Alcubierre *et.al.*, 2000; Yoneda & Shinkai, 2002; Lindblom & Scheel, 2002]

Other approaches

Recast ADM equations in first order symmetric hyperbolic form (see [Reula, 1998] for review).

Example: "Einstein-Christoffel" system

$$\frac{d}{dt}\gamma_{ij} = -2\alpha K_{ij}$$

$$\frac{d}{dt}K_{ij} + \alpha\gamma^{kl}\partial_l f_{kij} = \alpha M_{ij}$$

$$\frac{d}{dt}f_{kij} + \alpha\partial_k K_{ij} = \alpha N_{kij}$$

where f_{kij} are linear combinations of Christoffel symbols [Anderson & York, 1999; Kidder *et.al.*, 2001]

What else do we have to worry about?

- Have talked about principle part, but not yet about source terms...
- Choice of lapse α ("slicing condition")
- Choice of shift β^i ("spatial gauge condition")
- For black holes: deal with singularity
- For neutron stars: deal with matter
- Boundary conditions
- All kinds of computational issues...

Slicing conditions

Bad idea: geodesic slicing ($\alpha = 1$, $\beta^i = 0$): leads to coordinate singularities

Generally look for "symmetry-seeking" coordinate conditions

- "Maximal" slicing:

$$\partial_t K = 0$$

  - elliptic equation for α, computationally expensive
  - Can find "approximate maximal slicing" by recasting elliptic equation as parabolic equation

- Harmonic slicing:

$$\partial_t \alpha = -\alpha^2 K$$

  or $\alpha \propto \gamma^{1/2}$ (properties poorly understood)

- "Bona-Masso" slicing:

$$\partial_t \alpha = -\alpha^2 f(\alpha) K$$

- "1+log" slicing [Bona *et.al.*, 1995] :
  choose $f = 2/\alpha$, then

$$\alpha = 1 + \ln \gamma$$

  Some promising properties; "algebraic"

PS: Probably advantageous to impose condition on densitized lapse [Kidder *et.al.*, 2001; Sperhake *et.al.*, 2003]

Spatial gauge conditions

Spatial gauge conditions for shift β^i

- Zero shift: $\beta^i = 0$
- Minimal distortion [Smarr & York, 1978] :
  - minimizes time change of conformally related metric
  - elliptic equations for β^i
  - typically, only approximate solutions have been constructed [Shibata, 1999]
- "Gamma-freezing" [Alcubierre & Brügmann, 2001] :

$$\partial_t \bar{\Gamma}^i = 0$$

  - similar to minimal distortion

Dealing with black holes

Singularities have unfortunate effects for numerical code - have to be avoided

- Singularity avoiding coordinates:
  - “hold back” time evolution in proximity of singularity
  - leads to other coordinate pathologies

  $\Longrightarrow$ not very promising for long-time evolutions

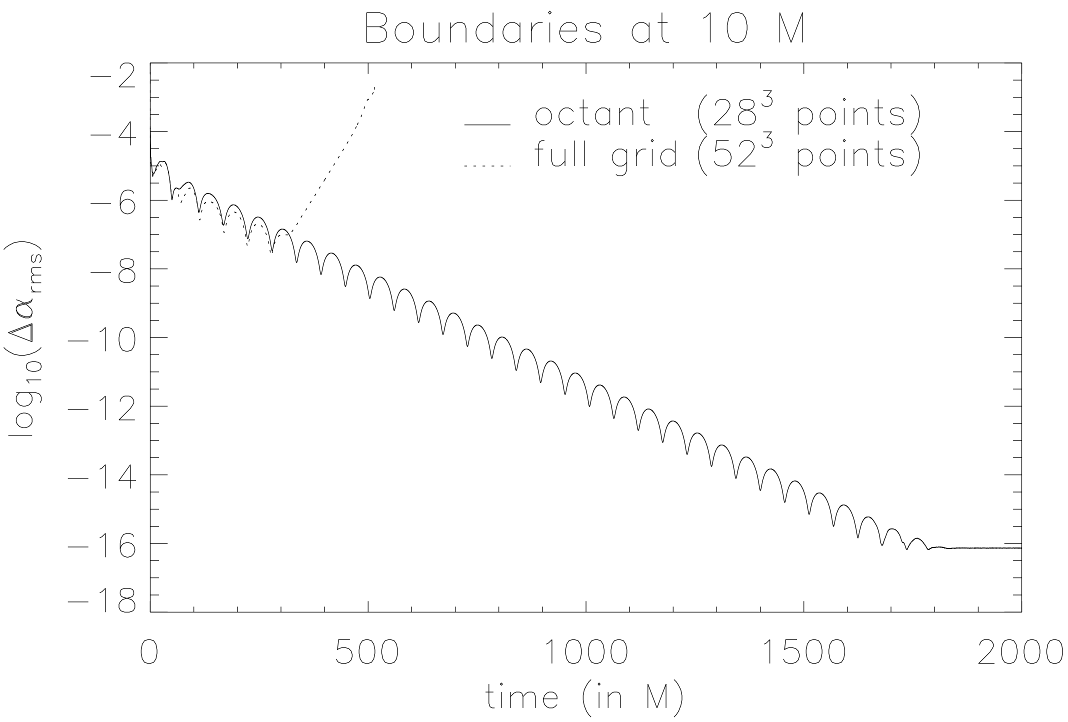
- “Puncture approach” [Brügmann, 1999] :
  - factor out singular parts analytically, compute only “corrections”
- Singularity excision [Unruh, 1984; Seidel & Suen, 1992] :
  - excise interior of black hole from computational grid
  - come up with suitable boundary conditions – simple copying seems fine!

Funny symmetry issue

Combine BSSN formulation with very simple excision for evolution of single black hole (in Kerr-Schild coordinates) [Alcubierre & Brügmann, 2001]

$\Longrightarrow$ stable when evolved in octant symmetry

$\Longrightarrow$ but unstable otherwise!?



[Alcubierre & Brügmann, 2001]

Dealing with source terms

- Have dealt with principal part
- Hyperbolicity guarantees that certain energy norms grow at most exponentially
- BUT: exponential growth not acceptable from numerical point of view

Imagine evolution equations contain expressions of form

$$\partial_t f = \ldots + af + \ldots$$

If $a > 0$ could find exponential growth

To eliminate af term or change its sign can...

- introduce new variables
- add constraint equations

$\Longrightarrow$ "adjustment" of equations

[Kidder, Scheel, Teukolsky, 2001; Shinkai & Yoneda, 2002; Scheel *et.al.*, 2002, Laguna & Shoemaker, 2002]

Adjusting the BSSN equations

In BSSN equations, $\partial_t \bar{\Gamma}^i$ equation contains such a term

$$\partial_t \bar{\Gamma}^i = ... + \frac{2}{3} \beta^j{}_{,j} \bar{\Gamma}^i$$

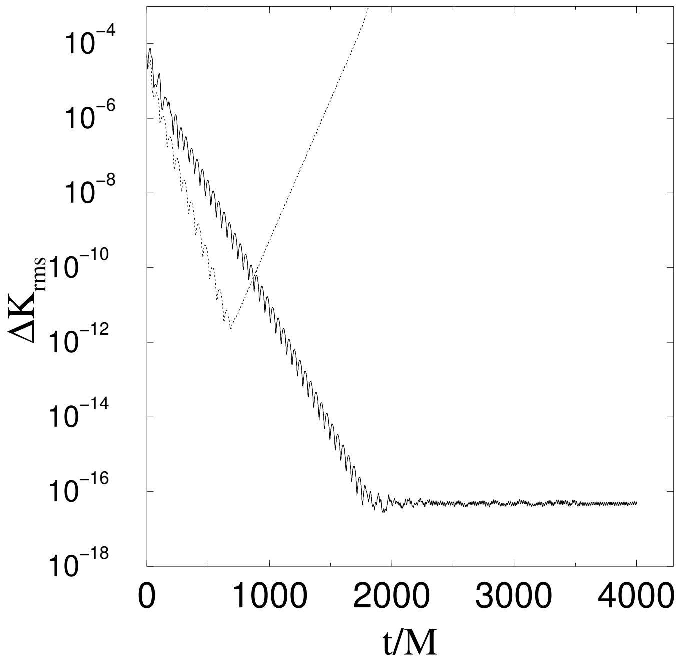
Sign can be changed easily by adding (algebraic) constraint

$$\mathcal{G}^i = \bar{\Gamma}^i - \bar{\gamma}^{lm} \bar{\Gamma}^i_{lm}$$

Then

$$\partial_t \bar{\Gamma}^i = \text{old RHS} - (\chi + \frac{2}{3}) \beta^j{}_{,j} \, \mathcal{G}^i$$

χ is free parameter, can be chosen dynamically



[Yo, Baumgarte & Shapiro, 2002]

Current status of black hole evolutions

- Single black holes:
  - Can evolve for thousands of M [Yo, Baumgarte & Shapiro, 2002; Sperhake *et. al.*, 2003; Scheel *et. al.*, 2002]
  - Some issues unresolved (e.g. outer boundaries)

- Binary black holes:
  - So far only crude prelimiary calculations [Brügmann, 1999; Brandt *et. al.*, 2000; Alcubierre *et. al.*, 2001; Baker *et. al.*, 2001]
  - Need suitable initial data (being constructed at Cornell/Caltech, Illinois...)
  - Dynamical codes under development at Penn State, Illinois, Caltech/Cornell...
  - Stay tuned!

Binary neutron stars

For evolution of neutron stars need to couple gravitational field equations to equations of relativistic hydrodynamics. See Font [2000] for review...

Can implement either...

- "easy": artificial viscosity scheme
  [Shibata, 1999; Duez, Marronetti, Shapiro & Baumgarte, 2003]
- "harder": high-resolution shock-capturing schemes
  [Font *et.al.*, 2000, 2002]

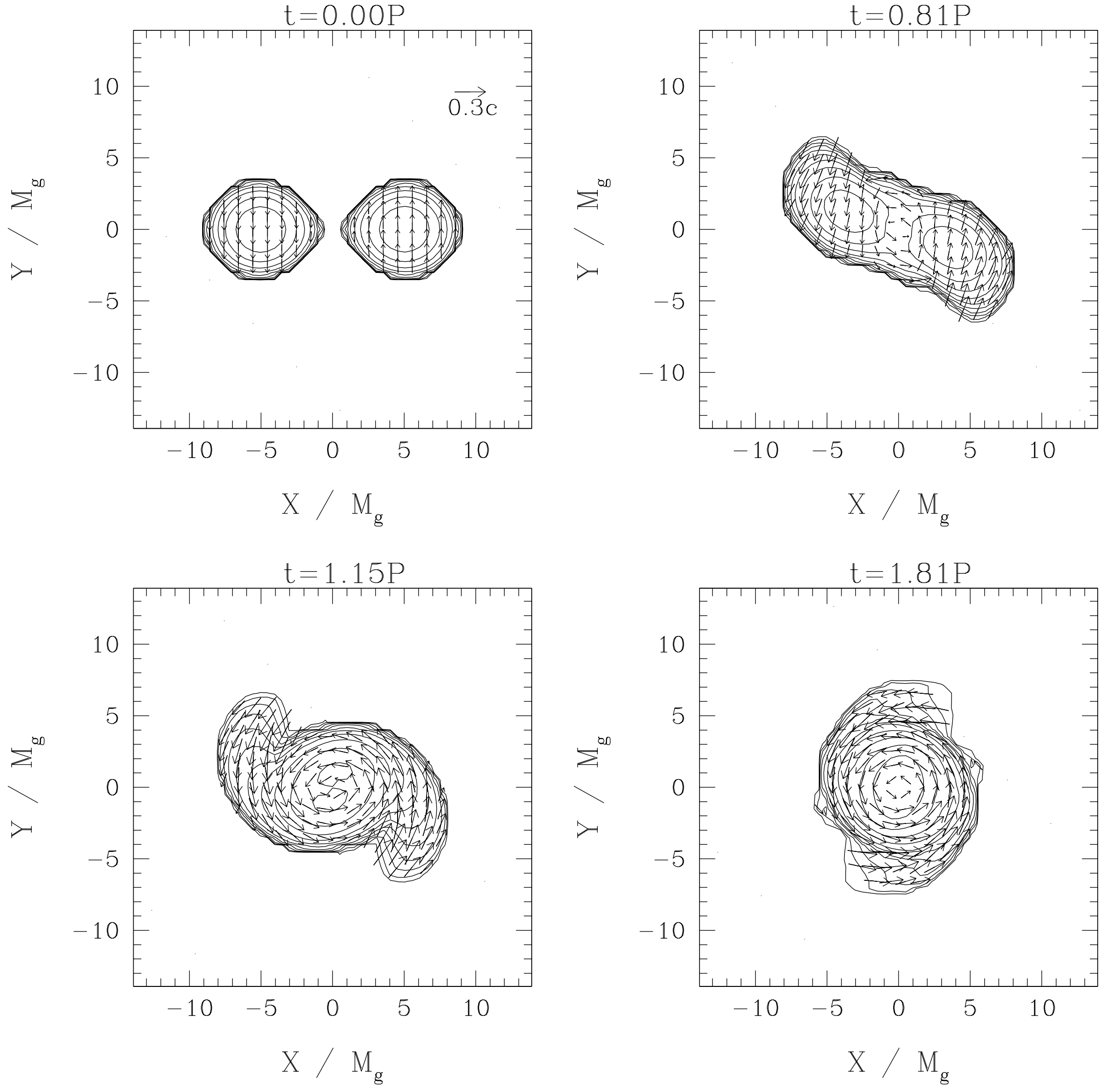
Simulations of binary neutron star mergers

First simulations of binary neutron star mergers by Shibata and Uryū [2000, 2002]

- evolve artificial viscosity scheme together with BSSN version of evolution equations for gravitational fields
- study several mergers of corotating and irrotational binaries of different mass

Example: irrotational binary, $n = 1$ polytrope, mass exceeds maximum allowed rest mass by 45%

Binary neutron star coalescence

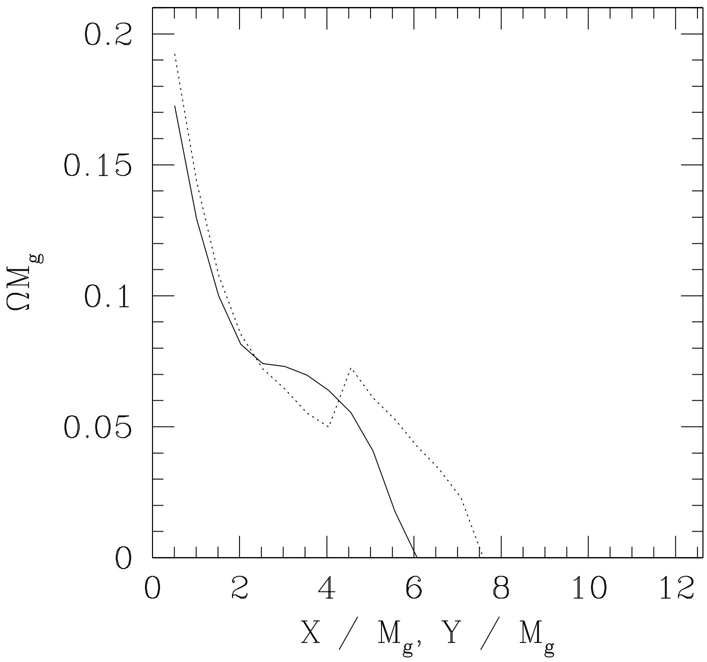


[Shibata & Uryū, 2000]

⟹ Surprising result: does *not* collapse to black hole

Effect of differential rotation

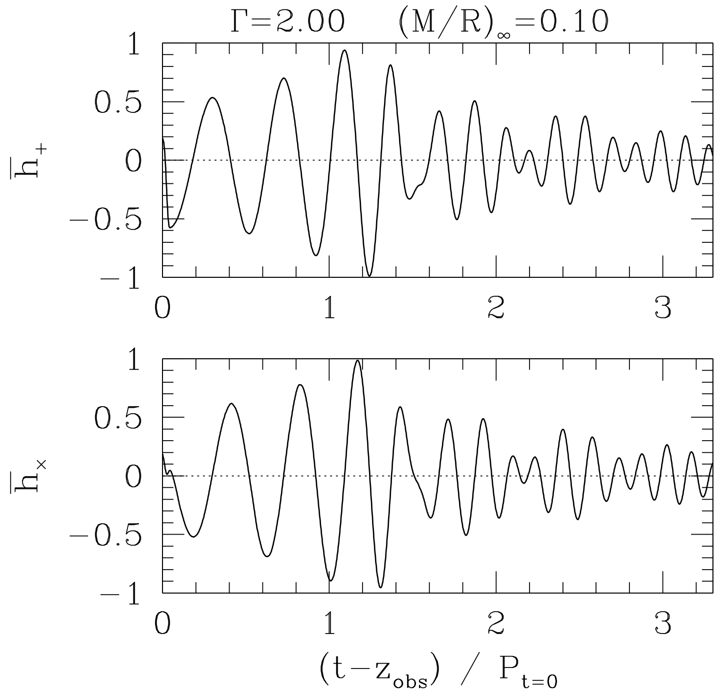
Remnant is *differentially* rotating:



[Shibata & Uryū, 2000]

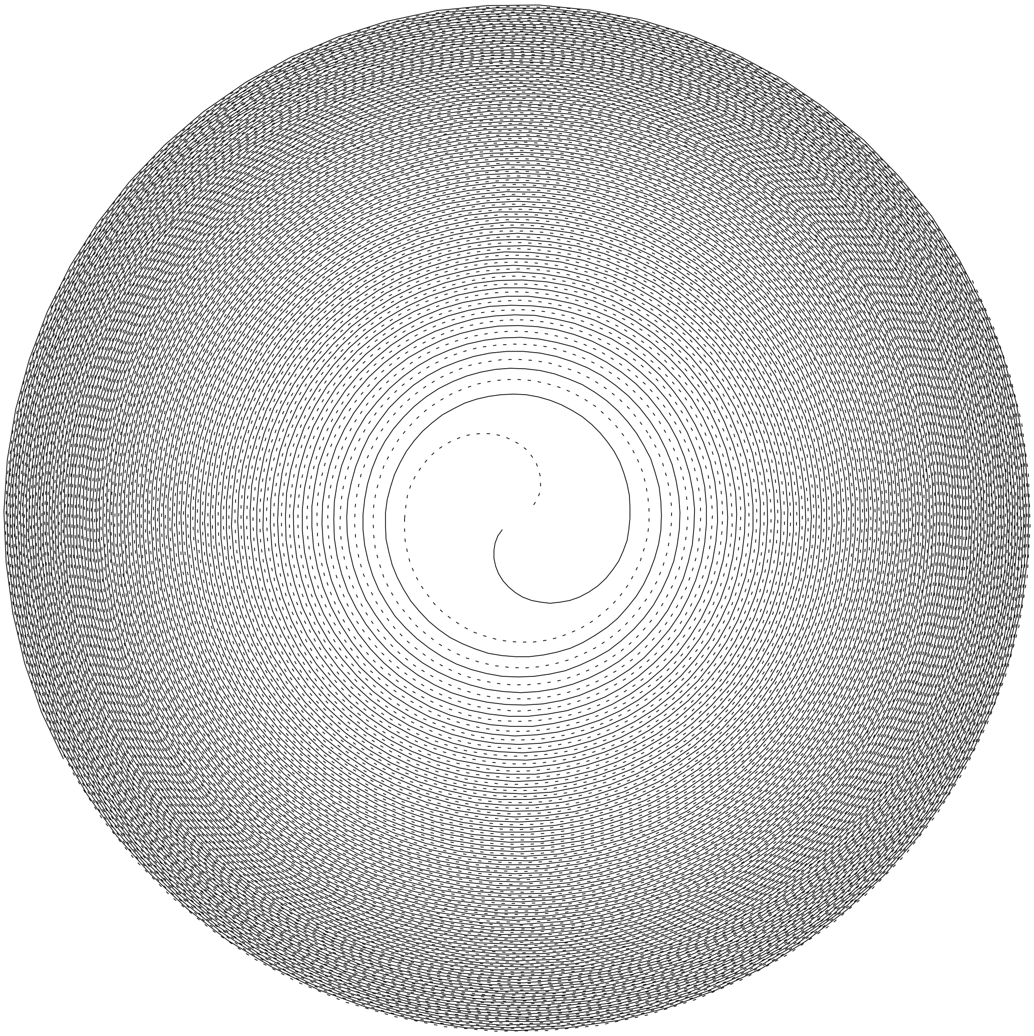
$\Longrightarrow$ Differential rotation can dramatically increase maximum allowed mass [Baumgarte, Shapiro & Shibata, 2000; Lyford, Baumgarte & Shapiro, 2003]

Gravitational wave form



[Shibata & Uryū, 2002]

Cartoon of binary inspiral



Quasiadiabatic inspiral

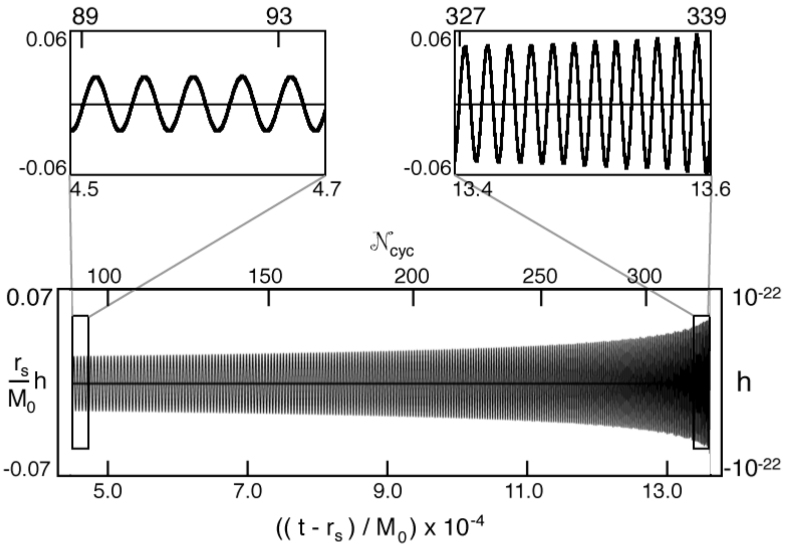
Can construct "quasi-adiabatic" inspiral sequences to model quasiadiabatic inspiral (prior to dynamical plunge and merger)

- Assume that matter distribution is adequately described by sequence of quasiequilibrium initial data
- Dynamically evolve only gravitational fields (in presence of predetermined matter sources)
  $\Longrightarrow$ Gravitational wave form and luminosity

- Inspiral rate can be computed from

$$\frac{dr}{dt} = \frac{dM/dt}{dM/dr}$$

$\Longrightarrow$ construct gravitational wave train from inspiral phase



[Duez *et. al.*, 2002]

Current status of binary neutron star calculations

- Can model both initial data and their evolution
- Would like to relax several approximations and assumptions, but no point of principle problem
- Simulations produce results of astrophysical interest