Focusing timelike or null singularity solutions

for a radiative spherically symmetric gravitational source

and cosmological applications

Marie-Noëlle Célérier

LUTH

Observatoire de Paris-Meudon

MNC & Jean Schneider, 1998, Phys. Lett. A249, 37
MNC & Peter Szekeres, 2002, Phys. Rev. D65, 123516

Mathematics of Gravitation II
September 2003

Motivations

1. The cosmological horizon problem.

   Look for a geometric alternative solution to the inflation hypothesis in Einstein's general relativity framework .

   First stage (collaboration with Jean Schneider):
   proposal of the "Delayed Big-Bang" model.

   Second stage (collaboration with Peter Szekeres):
   need to refine this model in the radiation dominated era.

2. The strong cosmic censorship hypothesis.

   Propose physically consistent counterexamples.

3. Complete a study initiated by
   P. Szekeres & V. Iyer, 1993, Phys. Rev. D47, 4362,
   using the method they proposed.

The cosmological horizon problem

In standard hot Big-Bang models, the comoving region of the last-scattering surface over which the cosmic microwave background radiation (CMBR) is observed to be homogeneous to better than one part in 10^5 is much larger than the intersection of this surface with any future light-cone from the "Big-Bang". As this light-cone provides the maximal distance over which causal processes could have propagated since a given point on the "Bang" singularity, the observed isotropy of the CMBR remains unexplained in the framework of these models and is known as the horizon problem.

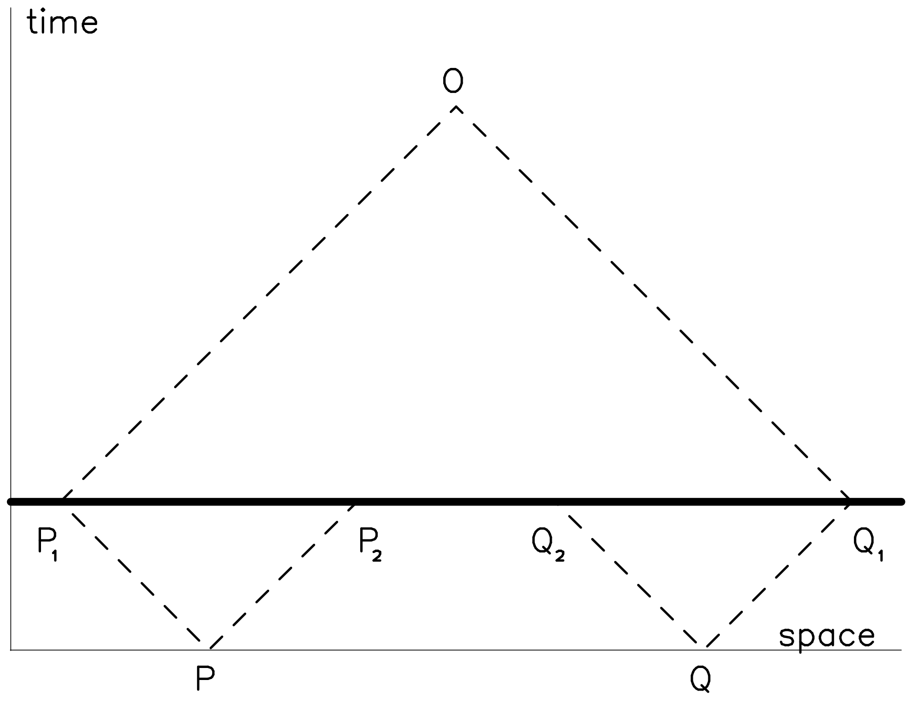

Figure 1. Conformal diagram.

The inflationary solution

The horizon problem develops sooner or later in any cosmological model exhibiting a spacelike singularity such as that occurring in standard FLRW universes.

Since it does not change the spacelike character of the singularity, inflation only postpones the occurrence of the problem. Adding an inflationary phase in the primordial history of the universe amounts to adding a slice of de Sitter spacetime. The effect is merely to postpone the event L beyond which the current observer O sees a causally connected last-scattering surface.

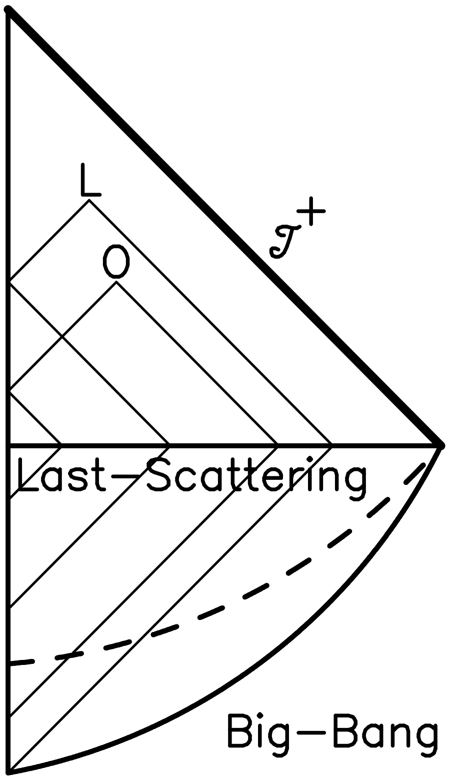

Figure 2. Penrose-Carter conformal diagram for a spacelike singularity universe.

Non spacelike singularity solution

A permanent solution to the problem is a model of universe exhibiting a timelike or null singularity. In this case, any observer, whatever his space-time location, sees in his CMBR a causally connected last scattering surface.

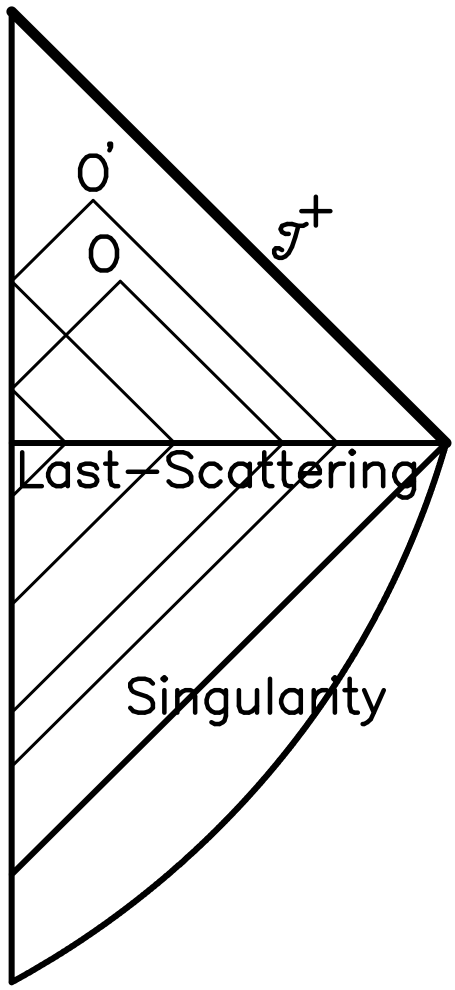

Figure 3. Penrose-Carter conformal diagram for a universe with a timelike or null singularity.

The delayed Big-Bang model

First proposal:

Delayed Big-Bang models of universe (DBB). The dust dominated region of space-time is represented by a spatially flat Lemaître-Tolman-Bondi (LTB) solution, with spherical symmetry. (MNC & Schneider, 1998).

We justify the spherical symmetry of the model using both arguments of the isotropy of the cosmological observations and of the possibility of averaging over the angular coordinates. Spatial flatness is assumed to be consistent with current observations.

Main feature of this model:

Exhibiting two singularities, one of the "shell cross" type, the other focusing (or "Big-Bang", with a vanishing area) and being constructed such as the shell cross singularity is situated between the last scattering surface and the Big-Bang. This implies that the "Bang" singularity $t = t_0(r)$, where $t_0(r)$ is an arbitrary function in this class of models, exhibits a $t_0(r)$ function monotonously increasing with r, therefore justifying the name "Delayed Big-Bang" given to these models: the cosmological (proper) time elapsed from the "Bang" singularity is shorter for a far off-center observer than for an observer located near the "center" of the model.

Schematic illustration of a DBB universe

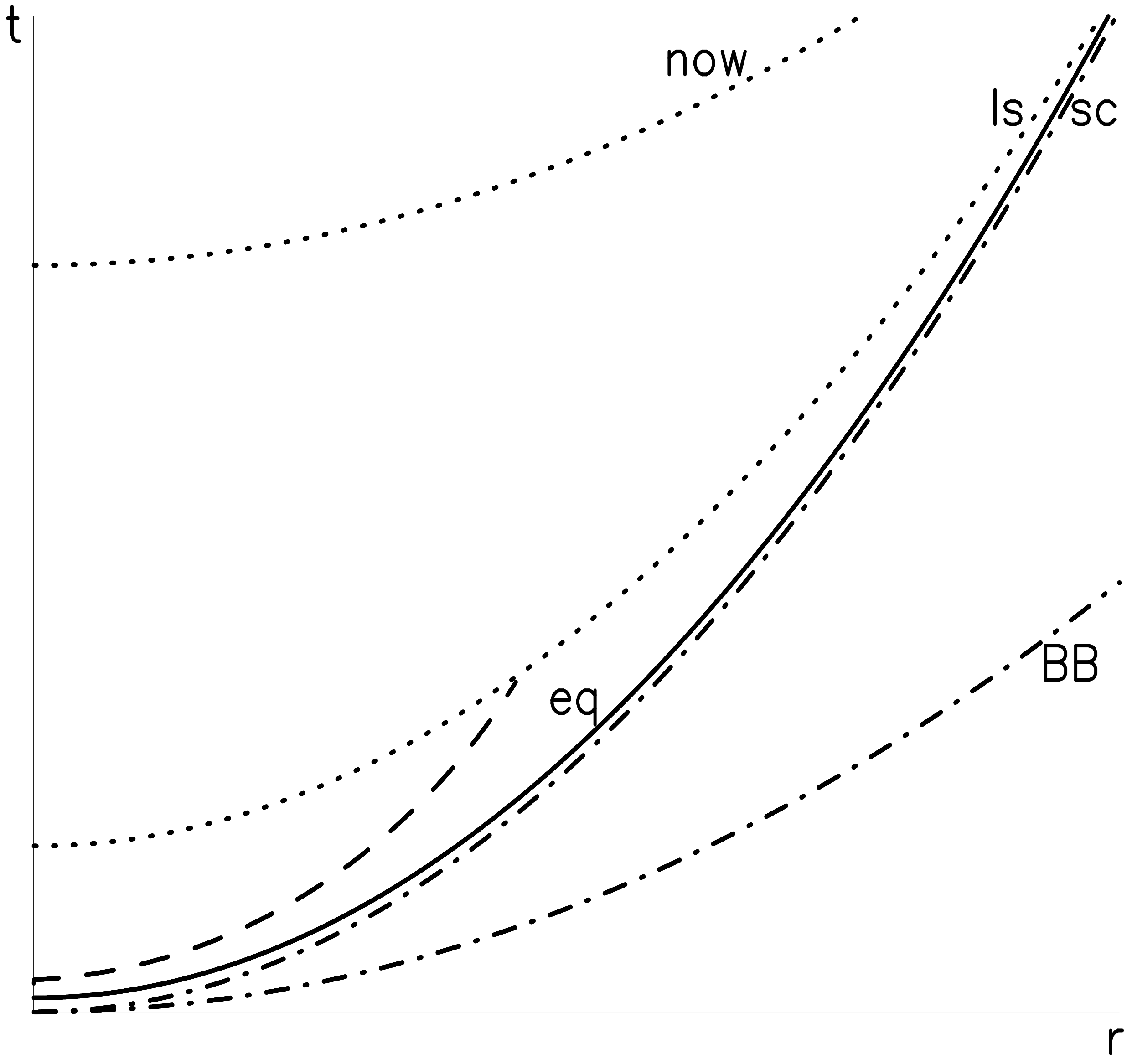

Figure 4. The dot-dash-dot lines represent the "Bang" and the shell-crossing singularity surfaces. The dotted lines are the constant energy-density (constant temperature) surfaces, i.e., the last scattering and "now" surfaces. The solid line is the radiation-matter equality surface.

Derivation of the timelike character of the shell-crossing singularity in a spherically symmetric model

Generalization of a line of reasoning first proposed by Hellaby & Lake, 1985.

The general spherically symmetric line element can be written

$$ds^2 = -B^2(r,t)dt^2 + A^2(r,t)dr^2 + R^2(r,t)(d\theta^2 + \sin^2\theta d\varphi^2). \quad (1)$$

A typical shell-crossing surface $t = b(r)$ is such that

$$A = [t - b(r)]^a f(r,t) = 0 \qquad B \neq 0 \qquad R \neq 0. \quad (2)$$

The normal, n_α, to the surface $A = \text{const.}$ (here $A = 0$), is

$$n_\alpha \propto (\dot{A}, A', 0, 0). \quad (3)$$

With the metric of Eq. (1), the squared norm of this normal vector is

$$n_\alpha n^\alpha = -\dot{A}^2 B^2 + A'^2 A^2. \quad (4)$$

According to Eq. (2), $A^2 = 0$, and the above expression for $n_\alpha n^\alpha$ is always negative. This implies, with our choice of the metric signature, a timelike shell-crossing surface.

Application to the DBB model

THEOREM:
Every spherically symmetric shell cross singularity is timelike.

The DBB model is therefore free of any horizon problem.

Drawback of the model:
When looking backward towards the cosmological past, the energy density increases and the radiative pressure can no more be neglected.

Solutions:
-either put constraints on the "Bang" function $t_0(r)$ of the DBB model, so that the backward light-cones issued from the last scattering surface reconnect before reaching the radiation-matter equality surface. The fastest the $t_0(r)$ function increases with r, the nearest from the last-scattering the light-cones reconnect. We can therefore set bounds on the $t_0(r)$ function to ensure light-cone reconnection before the radiation-matter equality surface (MNC & Schneider, 1998).

-or forget the DBB model and construct a radiative model with a non-spacelike singularity.

The cosmic censorship hypothesis

Cosmic censorship hypothesis (CCH): No physically realistic collapse (i.e., evolving from well posed initial data and satisfying the dominant energy condition) leads to a naked singularity (i.e. located in the causal past of points at future infinity). (Penrose, 1969)

If a singularity is naked, one could prescribe arbitrary data on the singularity surface. This would result in a total loss of predictability in the future of the singularity. If a singularity is hidden behind an event horizon, predictability is preserved, at least in the space-time region outside the horizon.

After the debate that followed this first proposal, which gave rise to a number of counter-examples, not always convincing, Penrose proposed the Strong Cosmic Censorship Hypothesis (SCCH): No physically realistic collapse leads to a locally naked singularity. (Penrose, 1979)

The singularity is locally naked if light-rays do emerge from the singularity but fall back to the center without escaping the event-horizon. Such a singularity will be visible to an infalling observer who has entered the horizon of the collapsing body, but cannot be seen by asymptotic observers.

A singularity is globally naked if light-rays emerging from the singularity escape the event-horizon, therefore possibly reaching an observer located outside this horizon.

The weak cosmic censorship hypothesis (CCH) allows for the occurrence of locally naked singularities but not globally naked ones, whereas the strong cosmic censorship (SCCH) does not allow either.

In a black hole model with a spacelike or null singularity, an observer falling inside the apparent horizon (the trapped surface envelope), which appears after the initial collapse, will reach the singularity before he can see it, i.e., his backward light-cone cannot cross the singularity surface.

But in a collapsing body with a timelike singularity, the infalling observer may be able to see the singularity before reaching it, even if all the light rays are trapped inside the final black hole (no naked singularity visible from infinity). But, from the point of view of infalling particles or observers, the local loss of predictability is as worrying as a naked singularity visible from infinity (and violates the SCCH).

A timelike singularity is thus a necessary and sufficient condition for SCCH violation.

Curvature strenght: A curvature singularity is called strong if collapsing volume elements do get crushed to zero at the singularity, and weak if they do not. A strong naked singularity is regarded as a more serious violation of censorship than a weak one. It is believed that space-time cannot be extended through a strong singularity, but is possibly extendible through a weak one. For instance, shell-crossing type singularities are gravitationally weak.

Spherically symmetric collapse with different singularity types

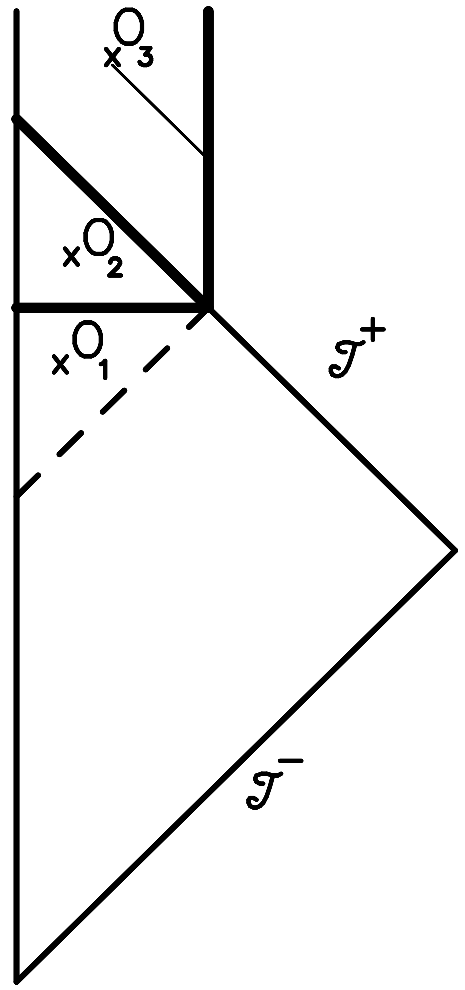

Figure 5. Penrose-Carter conformal diagram. The dashed line is the horizon. The solid thick lines represent the singularity which can be spacelike, null or timelike. The only observer falling inside the horizon of the collapsing body who can see the singularity is O_3, because the singularity is timelike.

The technical problem exposition

We are looking for solutions of the Einstein equations verifying:

the dominant energy condition, $\rho > |\mathrm{P}|$, with a peculiar interest for those exhibiting the radiative equation of state,

$$\mathrm{P} = \frac{1}{3}\rho,$$

in the vicinity of a timelike or null singularity.

Since we do not know exact solutions for such a gravitational source, we limit ourselves to:

spherically symmetrical solutions,
exhibiting "power-law type" singularities,
i.e., with a line element written as

$$\mathrm{ds}^2 = -\mathrm{dt}^2 + [\mathrm{t} - \tau(\mathrm{r})]^{2\mathrm{a}} \mathrm{f}^2(\mathrm{r}, \mathrm{t}) \mathrm{dr}^2 + [\mathrm{t} - \tau(\mathrm{r})]^{2\mathrm{b}} \mathrm{g}^2(\mathrm{r}, \mathrm{t}) \mathrm{d}\Omega^2, \quad (5)$$

where f and g are functions of r and t which are regular and non-vanishing at the singularity surface $\mathrm{t} = \tau(\mathrm{r})$.
and which we study in the singularity neighborhood,
with the method developed by Szekeres & Iyer, 1993.

Summary of the mathematical method

We use the double null coordinates $u(r,t)$ and $v(r,t)$ to write the singularity equation under the form $t = \tau(r)$ and $x = 0$, with $x = lu + kv$ and $l, k = \pm 1$ or 0.

This singularity is spacelike if $lk = 1$, timelike if $lk = -1$ and null if $lk = 0$.

We expand the metric coefficients in the neighborhood of the singularity, which yields the non zero components of the Einstein tensor G^{μ}_{ν} as power of x series expansions.

The Einstein equations with a perfect gas with anisotropic pressure as a gravitational source give relations between the energy density, the pressure components and the G^{μ}_{ν}'s.

Adding the dominant energy condition imposes a new series of constraints upon these same quantities.

Constraints upon the exponents of the series expansion follow, the selfcompatibility of which we examine for each possible value of the lk product: +1, -1 and 0. The retained solutions are those for which these sets of constraints are consistent.

Szekeres & Iyer, 1993, had studied the cases $kl \neq 0$ (timelike or null singularity) and $q < p + 2$. MNC & Szekeres, 2002, analysed the remaining cases.

A brief description of the mathematical method

We pass to **double null coordinates** $u(r,t)$ and $v(r,t)$, such that

$$ds^2 = -2e^U dudv + e^V d\Omega^2. \tag{6}$$

In this coordinate system, the radial light-cones ($\theta = \text{const}$, $\phi = \text{const}$) verify equation $u = \text{const}$ or $v = \text{const}$.

The coordinate transformation is done in the vicinity of the shell $r = r_0$, $t = \tau(r_0)$ on the singularity surface, by carrying out a series expansion of the form:

$$\begin{aligned}
r &= r_0 + u + f_1(u)x^{a_1} + f_2(u)x^{a_2} + \dots & (7)\\
t &= \tau(r_0 + u) + g_1(u)x^{b_1} + g_2(u)x^{b_2} + \dots & (8)\\
&= \tau(r_0) - \tau'(r_0 + u)f_1(u)x^{a_1} + g_1(u)x^{b_1} + \dots & (9)
\end{aligned}$$

with $0 < a_1 < a_2 < \dots$, and $0 < b_1 < b_2 < \dots$.

By Eq. (9), **the singularity** at $t = \tau(r)$ **occurs at $x = 0$**, and the freedom available in the definition of the u and v coordinates can be used to express the function $x(u,v)$ in the form

$$x = lu + kv \qquad \text{with} \qquad l, k = \pm 1 \text{ or } 0. \tag{10}$$

For the cosmological expansion case, the signs of l and k are chosen such that $x > 0$ for $t > \tau(r)$ in the neighborhood of the singularity, the reverse for the collapsing case.

This singularity $x = 0$ is **spacelike if $lk = 1$**, **timelike if $lk = -1$**, and **null if $lk = 0$**.

We show that the functions U and V can be written

$$e^U = 2t_u t_v = x^p e^\alpha, \qquad e^V = (t-\tau)^{2b} g^2 = x^q e^\beta \tag{11}$$

$$\text{with} \qquad \alpha = \alpha_0(u) + \alpha_1(u) x^{p_1} + \dots, \tag{12}$$

$$\text{and} \qquad \beta = \beta_0(u) + \beta_1(u) x^{q_1} + \dots, \tag{13}$$

The exponents p and q depend on a and b, and the $p_1, p_2, \dots, q_1, \dots$ on $a_1, a_2, \dots$ $b_1, b_2, \dots$ appearing in Eqs. (7-9).

The nonvanishing components of the Einstein tensor for the metric (6), with $x^0 = u$, $x^1 = v$, $x^2 = \theta$, $x^3 = \phi$, are

$$G_0^0 = G_1^1 = -x^{-q} e^{-\beta} - e^{-\alpha} x^{-p} \Big(\frac{kl(q^2 - q)}{x^2} + \frac{k\beta_u + l\beta_v}{x} + \beta_{uv} + \beta_u \beta_v \Big) \tag{14}$$

$$G_1^0 = -e^{-\alpha} x^{-p} \Big(\frac{k^2(q + pq - \frac{1}{2}q^2)}{x^2} + \frac{k((p-q)\beta_v + q\alpha_v)}{x} - \beta_{vv} + \alpha_v \beta_v - \frac{1}{2}\beta_v^2 \Big) \tag{15}$$

$$G_0^1 = -e^{-\alpha} x^{-p} \Big(\frac{l^2(q + pq - \frac{1}{2}q^2)}{x^2} + \frac{l((p-q)\beta_u + q\alpha_u)}{x} - \beta_{uu} + \alpha_u \beta_u - \frac{1}{2}\beta_u^2 \Big) \tag{16}$$

$$G_2^2 = G_3^3 = -e^{-\alpha} x^{-p} \Big(\frac{lk(\frac{1}{2}q^2 - p - q)}{x^2} + \frac{q(k\beta_u + l\beta_v)}{2x} + \alpha_{uv} + \beta_{uv} + \frac{1}{2}\beta_u \beta_v \Big) \tag{17}$$

Notations: $\alpha_u \equiv \frac{\partial \alpha}{\partial u}$, $\alpha_{uv} \equiv \frac{\partial^2 \alpha}{\partial u \partial v}$, etc.

Inside the stress-energy tensor, the pressure is decomposed into its radial P_r and transverse $P_\perp$ components:

$$T^\mu_{\ \nu} = \rho u^\mu u_\nu + P_r f^\mu f_\nu + P_\perp h^\mu_{\ \nu}, \tag{18}$$

The Einstein equations therefore give

$$\rho = -G^0_0 - G^0_1\sqrt{\frac{G^1_0}{G^0_1}},$$
$$P_r = G^0_0 - G^0_1\sqrt{\frac{G^1_0}{G^0_1}}, \qquad P_\perp = G^2_2 = G^3_3. \tag{19}$$

For ρ and P_r to be real, G^1_0 and G^0_1 must be both positive or negative.
The dominant energy condition $\rho > |P|$ implies

$$G^0_1 < 0, \qquad G^1_0 < 0. \tag{20}$$

$$G^0_0 < 0, \qquad |G^2_2| < -G^0_0 + \sqrt{G^0_1 G^1_0}, \tag{21}$$

which garantee a positive energy density and impose a radial pressure,

$$\rho = -G^0_0 + \sqrt{G^0_1 G^1_0}, \qquad P_r = G^0_0 + \sqrt{G^0_1 G^1_0}. \tag{22}$$

and for a perfect fluid whith a radiative equation of state, $P = P_r = P_\perp = \frac{1}{3}\rho$,

$$G^2_2 = -G^0_0, \qquad \sqrt{G^0_1 G^1_0} = -2G^0_0. \tag{23}$$

Constraints on the p and q parameters follow.
Equations (14-16) and (22) imply that if $\rho \to \infty$ when $x \to 0$, then $q > 0$ or $p > -2$. If the pressure is non-extreme in this limit, $P_r > -\rho$ when $x \to 0$ and $q \leq p + 2$, which give

$$p > -2, \qquad q \leq p + 2.$$

For each of the possible values of lk, $+1, -1$ et 0, we shall examine the consistency of these constraints with the dominant energy condition , before specializing our study to the radiation equation of state.

Illustration: the case $lk = -1, q = p + 2, p > -2$

This case implies $q > 0$, therefore the behavior of G^μ_ν, for $x \to 0$, is dominated by

$$\begin{aligned} G^0_0 &\approx -x^{-q}(e^{-\beta_0} - e^{-\alpha_0}(q^2 - q)), && (24)\\ G^1_0 &\approx G^0_1 \approx -x^{-q}e^{-\alpha_0}\frac{1}{2}q(q-2), && (25)\\ G^2_2 &\approx x^{-q}e^{-\alpha_0}\frac{1}{2}(q-2)^2. && (26) \end{aligned}$$

The constraints of Eq. (20), $G^0_1 < 0, G^1_0 < 0$, and $q > 0$ applied to Eq. (25) imply

$$q > 2.$$

The substitution of Eqs. (24-26) in Eq. (19) results in

$$\begin{aligned} \rho &\approx x^{-q}(e^{-\beta_0} - e^{-\alpha_0}\frac{q^2}{2}),\\ P_r &\approx x^{-q}(-e^{-\beta_0} + e^{-\alpha_0}\frac{q(3q-4)}{2}),\\ P_\perp &\approx x^{-q}e^{-\alpha_0}\frac{(q-2)^2}{2} > 0. \end{aligned}$$

The positive energy density condition, $\rho > 0$, implies

$$e^{-\beta_0} > e^{-\alpha_0}\frac{q^2}{2} \qquad (27)$$

which we write

$$e^{-\beta_0} = e^{-\alpha_0}(\frac{q^2}{2} + C_0(u)) \qquad \text{with} \qquad C_0(u) > 0, \qquad (28)$$

which gives

$$\rho \approx x^{-q} e^{-\alpha_0} C_0, \tag{29}$$

$$P_r \approx x^{-q} e^{-\alpha_0} (-C_0 + q^2 - 2q), \tag{30}$$

$$P_\perp \approx x^{-q} e^{-\alpha_0} \frac{(q-2)^2}{2}. \tag{31}$$

Any sensible equation of state can therefore be obtained in the vicinity of the singularity $x = 0$ by a judicious choice of the function $C_0(u)$.

For example, in the case of a perfect fluid (isotropic pressure):

$$P_r = P_\perp \implies C_0(u) = \frac{q^2}{2} - 2 > 0, \tag{32}$$

and

$$\gamma = \frac{P}{\rho} = \frac{q-2}{q+2} > 0. \tag{33}$$

A radiative equation of state $\gamma = \frac{1}{3}$ is achieved if

$$q = 4 \implies p = 2$$

and

$$C_0(u) = 6.$$

Conclusion of the study of the different possible cases

For a spherically symmetric solution,

a singularity surface of power-law type,

exhibiting in its vicinity the stress-energy tensor of a radiative isotropic perfect fluid with equation of state $P = \frac{1}{3}\rho$,

is timelike if $q = 4$ and $p = 2$,

null if $q = p + 1$ and $p > -1$.

The magnitude of the singularity area is 4π times the coefficient of $d\Omega^2$ in the expression for the metric. If this area vanishes, the singularity can be considered as a central focus (a “Big-Bang” in cosmology). If this area is finite, the singularity is usually regarded as being a shell cross.

In Eq. (6), the coefficient of $d\Omega^2$ is e^V, which, from the definition retained in Eq. (11), is equal to $x^q e^\beta$. Therefore, every singularity $x = 0$ such that $q > 0$ is a central focus.

This is the case for both the timelike and null singularity solutions identified above.

Cosmological applications

We are now provided with two different final solutions to the horizon problem, using the scheme of Fig. 3:

1. Assume that a consistent approximation of the dust dominated region of the universe can be a model pertaining to the DBB class and of which the parameters preserve causality on the last scattering surface. (MNC & Schneider, 1998).

2. Or take advantage of the above new results to suggest that in a radiation dominated primordial universe timelike or null singularities can occur, getting rid of any horizon problem, whatever the properties of the dust dominated era to come.

Other consideration derived from the results obtained in the special cases studied by Szekeres & Iyer, 1993:
A spherically symmetric timelike singularity of power-law type, which is not a shell-cross, and in whose neighborhood the stress-energy tensor satisfies the dominant energy condition, must either have an asymptotically extreme equation of state ($|P_r| \sim \rho$ or $|P_\perp| \sim \rho$), or possess in its vicinity a negative pressure ($P_r < 0$ or $P_\perp < 0$).

Now, a negative pressure is characteristic of a cosmological constant dominated universe. Such a universe would not be likely to exhibit any horizon problem.

Application to the cosmic censorship hypothesis

Reminder of the strong cosmic censorship hypothesis (SCCH):

No physically realistic collapse leads to a locally naked, (i.e., timelike) singularity.

We have identified above a class of timelike focusing singularities, of power-law type, with spherical symmetry, in the vicinity of which the gravitational source exhibits the stress-energy tensor of a radiative perfect fluid.

This class of solutions of the Einstein equations represents, to a good approximation, a spherical cloud of collapsing gas near its focusing point.

It therefore constitutes an interesting physically consistent counterexample to the SCCH, of the strong curvature type.

Conclusion

We have analysed the properties of metrics with spherical symmetry, satisfying the dominant energy condition and of which the singularities are of power-law type.

We have identified two classes of peculiar interest:

1. A timelike class exhibiting in the neighborhood of its focusing singularity the stress-energy tensor of a perfect fluid with the equation of state of radiation ($p = \frac{1}{3}\rho$).

2. A set of null classes, verifying identical properties.

We have considered two important applications of these results:

1. In cosmology, the possibility of solving the horizon problem.

2. In gravitational collapse, a physically consistent counterexample to the strong cosmic censorship hypothesis.