Skeleton Hamiltonian dynamics

Mathematics of Gravitation II
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Outline

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- A Wilson-like approach

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- Standard Hamiltonian description of N black holes

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- Application to circular motion

A WILSON-LIKE APPROACH

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$=$ 3-metric γ_{ij} conformally flat

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Objective: finding an analytical counterpart of Wilson

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- Can be derived from a Hamiltonian
  $\longrightarrow$ investigations performed in the ADM framework

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- supported by the “effacement” principle

STANDARD ADM DESCRIPTION

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- (x^i_A, p_{Ai}) = matter conjugate variables
- (γ_{ij}, π^{ij}) = field conjugate variables

Hamiltonian for N point-masses

$$H(x^i_A, p_{Ai}, \gamma_{ij}, \pi^{ij}) = \frac{c^4}{16\pi G}\left[\int d^3\mathrm{x}\,\left(N\mathcal{H} + N^i\mathcal{J}_i\right)\right.$$
$$\left. + \int d^3\mathrm{x}\,\partial_i(\partial_j\gamma_{ij} - \partial_i\gamma_{jj})\right]$$

with $\mathcal{H} = -\sqrt{\gamma}\mathrm{R} + \frac{1}{\sqrt{\gamma}}\left(\pi^i_{\ j}\pi^j_{\ i} - \frac{1}{2}(\pi^i_{\ i})^2\right)$
$$+\frac{16\pi G}{c^2}\sum_{A=1}^{N}\left(m_A^2 + \frac{p_{Ai}p_{Ai}}{c^2}\right)^{\frac{1}{2}}\delta_A$$

[super-Hamiltonian]

and $\mathcal{J}_i = -2\partial_j\pi^j_{\ i} + \pi^{kl}\partial_i\gamma_{kl} - \frac{16\pi G}{c^3}\sum_{A=1}^{N} p_{Ai}\delta_A$

[supermomentum]

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- decomposition of the field variables
  - decomposition of γ_{ij} for a conformal factor Ψ^4:

$$\gamma_{ij} = \Psi^4 \delta_{ij} + h_{ij}^{\mathrm{TT}} \text{ with } \partial_j h_{ij}^{\mathrm{TT}} = 0$$

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  - field momentum decomposition

    $$\pi^{ij} = \mathrm{STF}(2\partial_i \pi^j) + \pi^{ij}_{\mathrm{TT}} \text{ with } \partial_j \pi^{ij}_{\mathrm{TT}} = 0$$

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- New conjugate variables h_{ij}^{TT} and π^{ij}_{TT}
- Final Hamiltonian

$$\begin{aligned} H(x_A^i, p_{Ai}, h_{ij}^{\mathrm{TT}}, \pi^{ij}_{\mathrm{TT}}) &= \frac{c^4}{16\pi G}\int d^3\mathrm{x}\;\partial_i(\partial_j\gamma_{ij} - \partial_i\gamma_{jj}) \\ &= -\frac{c^4}{2\pi G}\int d^3\mathrm{x}\;\Delta\Psi \end{aligned}$$

EFFECTIVE SKELETON HAMILTONIAN

1^{st} ingredient of the model

$\gamma_{ij} \propto \delta_{ij}$:

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lose of accuracy if v/c large *but* good in the static limit

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- $\pi^{ij} = \gamma^{ik} \pi^j_k = \Psi^{-4} \pi^i_j$
  $\Rightarrow \pi^i_i = 0$ (maximal slicing)
  π^i_j STF (symmetric trace-free)

Constraints equations for $h_{ij}^{\mathrm{TT}} = 0$

$$\Delta\Psi = -\frac{1}{8\Psi^7}\pi^i{}_j\pi^j{}_i - \frac{2\pi G}{c^2}\sum_{A=1}^{N} m_A\delta_A\Psi^{-1}\left[1 + \frac{p_{Ai}p_{Ai}}{m_A^2c^2\Psi^4}\right]^{\frac{1}{2}}$$

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regular at $r_A = |\mathbf{x} - \mathbf{x}_A| = 0$ for appropriate dimensions
agreement with Bowen & York

Flesh and skeleton

$$\Psi^{-7}\pi^i{}_j\pi^j{}_i = \Psi^{-7}\pi^i{}_j\mathrm{STF}(2\partial_i V_j)$$
$$= \Psi^{-7}\big(-2V_i\partial_j\pi^i{}_j + 2\partial_j(V_i\pi^i{}_j)\big) = \mathcal{O}\left(\frac{1}{c^6}\right)$$

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Flesh and skeleton

$$\Psi^{-7}\pi^i_{\ j}\pi^j_{\ i} = \Psi^{-7}\pi^i_{\ j}\mathrm{STF}(2\partial_i V_j)$$
$$= \Psi^{-7}\big(\underbrace{-2V_i\partial_j\pi^i_{\ j}}_{\text{skeleton}} + \underbrace{2\partial_j(V_i\pi^i_{\ j})}_{\text{flesh}}\big) = \mathcal{O}\left(\frac{1}{c^6}\right)$$

Properties of the flesh terms

- corresponds to a 2PN effect in γ_{ij} and H
- vanishes when $p_{Ai} = 0$
- vanishes in the test body limit
- contains poles in dimensional regularisation
  $\leftarrow$ compensated by poles in h^{TT}_{ij} in Einstein theory

3rd ingredient: $2\partial_j(V_i\pi^i{}_j) \to 0$

elimination of the flesh term

$$\Psi^{-7}\pi^i{}_j\pi^j{}_i \to \Psi^{-7}\big(-2V_j\partial_i\pi^i{}_j\big) = \frac{16\pi G}{c^3}\sum_{A=1}^{N}\Psi^{-7}_{\mathbf{x}=\mathbf{x}_A}p_{Aj}V_j\delta_A$$

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structure of Ψ

$$\begin{cases} \Delta\Psi = \sum_A f_A(\mathrm{x})\delta_A \\ \Psi = 1 + \mathcal{O}\left(\frac{1}{r}\right) \text{ at spatial infinity} \end{cases} \Rightarrow \quad \Psi = 1 + \sum_{A=1}^{N}\frac{G\alpha_A}{2r_A c^2}$$

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skeleton super-Hamiltonian constraint

$\hookrightarrow$ obtained by inserting ϕ into $\mathcal{H}$

Hamiltonian constraint

$$-\frac{c^2}{2\pi G}\Delta\Psi = \sum_{A=1}^{N} \alpha_A \delta_A$$

$$= \sum_{A=1}^{N} \Bigg\{ m_A \left(1 + \sum_{B\neq A} \frac{G\alpha_B}{2r_{AB}c^2}\right)^{-1} \left[1 + \left(1 + \sum_{C\neq A} \frac{G\alpha_C}{2r_{AC}c^2}\right)^{-4} \frac{p_{Ai}p_{Ai}}{m_A^2 c^2}\right]$$

$$+ \left(1 + \sum_{B\neq A} \frac{G\alpha_B}{r_{AB}c^2}\right)^{-7} \frac{p_{Ai}V_{Ai}}{c} \Bigg\} \delta_A$$

Equation for the lapse

$$\begin{cases} \text{evolution of } \pi^{ij} \text{ in a (3+1) splitting: } \partial_t \pi^{ij} = F^{ij}[\gamma_{kl}, N_k, x_A^k, p_{Ak}] \\ \text{gauge condition: } \pi^{kk} = 0 \\ \text{useful formulae: } D_i D_i N = 2\partial_i N \partial_i \ln \Psi + \Delta N, \ \mathrm{R}\sqrt{\gamma} = -8\Psi\Delta\Psi \end{cases}$$

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$$\Downarrow$$

$$N\Delta\Psi + \Delta\left[\Psi N\right] = \frac{3}{4}\Psi^{-7} N \pi^i_{\ j}\pi^j_{\ i} + \frac{4\pi G}{c^4}\Psi^{-5} N \sum_{A=1}^{N} \delta_A \frac{p_{Ai}p_{Ai}}{m_A}\left(1 + \frac{p_{Ai}p_{Ai}}{m_A^2 c^2 \Psi^4}\right)^{1/2}$$

Equation for the shift function

Evolution equation for γ_{ij}

$$\frac{1}{c}\partial_t \gamma_{ij} = \frac{N}{\sqrt{\gamma}}(2\pi_{ij} - \pi^k_k \gamma_{ij}) + 2D_{(i}N_{j)}$$

with $\quad 2D_{(i}N_{j)} = \gamma_{ki}\partial_j N^k + \gamma_{kj}\partial_i N^k + N^k \partial_k \gamma_{ij}$

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Conformal flatness

$$\gamma_{ij} - \frac{1}{3}\gamma_{kk}\delta_{ij} = 0 \quad \Longrightarrow \quad \partial_t\left(\gamma_{ij} - \frac{1}{3}\gamma_{kk}\delta_{ij}\right) = 0$$

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$$\text{STF}(\partial_i N^j) = -\Psi^{-6} N \pi^i_{\ j}$$

in particular: $\quad \Delta N^i + \frac{1}{3}\partial_{ij}N^j = -2\partial_j\left(\Psi^{-6}N\pi^i_{\ j}\right)$

Lapse function

Auxiliary quantities natural to pose: $N = \chi/\Psi$ with $\lim\limits_{|\mathbf{x}|\to+\infty} N = 1$

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$$\chi = 1 - \sum_{A=1} \frac{G\beta_A}{2r_A c^2}$$

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Elimination of the flesh term

$$\Psi^{-1}\chi\Delta\Psi + \Delta\chi = \frac{3}{4}\Psi^{-8}\chi\pi^i{}_j\pi^j{}_i$$
$$+\frac{4\pi G}{c^4}\Psi^{-6}\chi\sum_{A=1}^{N}\delta_A\frac{p_{Ai}p_{Ai}}{m_A}\left(1+\frac{p_{Ai}p_{Ai}}{m_A^2c^2\Psi^4}\right)^{1/2}$$

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Elimination of the flesh term

replaced by $\frac{12\pi G}{c^3}\sum_{A=1}^{N}\left(\frac{\chi}{\Psi^8}\right)_{\mathbf{x}=\mathbf{x}_A} p_{Aj} V_j \delta_A$

$$\Psi^{-1}\chi\Delta\Psi + \Delta\chi = \frac{3}{4}\Psi^{-8}\chi\pi^i{}_j\pi^j{}_i$$

$$+\frac{4\pi G}{c^4}\Psi^{-6}\chi\sum_{A=1}^{N}\delta_A\frac{p_{Ai}p_{Ai}}{m_A}\left(1+\frac{p_{Ai}p_{Ai}}{m_A^2c^2\Psi^4}\right)^{1/2}$$

Skeleton Hamiltonian for 2 black holes

equations satisfied by $\Psi_1 \equiv \Psi_{\mathrm{x}=\mathrm{x}_1}$ and Ψ_2
deduced from the linear independence of the δ_A's

$$\Psi_1 = 1 + \frac{Gm_2}{2r_{12}c^2\Psi_2}\left(1 + \frac{p_2^2}{m_2^2c^2\Psi_2^4}\right)^{\frac{1}{2}} + \frac{Gp_{2i}V_{2i}}{2r_{12}c^3\Psi_2^7}$$

$$\Psi_2 = 1 + \frac{Gm_1}{2r_{12}c^2\Psi_1}\left(1 + \frac{p_1^2}{m_1^2c^2\Psi_1^4}\right)^{\frac{1}{2}} + \frac{Gp_{1i}V_{1i}}{2r_{12}c^3\Psi_1^7}$$

where $\quad \Psi_1 = 1 + \frac{G\alpha_2}{2r_{12}c^2} \quad$ and $\quad r_{12} = |\mathrm{x}_1 - \mathrm{x}_2|$

$$H = -\frac{c^4}{2\pi G}\int d^3\mathrm{x}\, \Delta\Psi = c^2(\alpha_1 + \alpha_2)$$

Skeleton field for 2 black holes I

3-metric

$$\gamma_{ij} = \Psi^4 \delta_{ij} \quad \text{with} \quad \Psi = 1 + \frac{G}{2c^2}\left(\frac{\alpha_1}{r_1} + \frac{\alpha_2}{r_2}\right)$$

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Lapse function

obtained by projection of the lapse equation on δ_A

$$\boxed{\begin{aligned} \chi_B = 1 - \frac{G m_A}{r_{12} c^2} \Psi_A^{-4} \chi_A \Bigg\{ & \frac{7 p_{Ai} (V_{i\mathbf{x}=\mathbf{x}_A})}{m_A c} \\ & + \left[1 + \frac{p_{Ai} p_{Ai}}{m_A^2 c^2 \Psi_A^4}\right]^{-1/2} \left[3 \Psi_A^2 \frac{p_{Ai} p_{Ai}}{m_A^2 c^2} + \Psi_A^6\right] \Bigg\} \\ \text{where} \quad \chi_1 = 1 - \frac{G \beta_2}{2 r_{12} c^2} & \end{aligned}}$$

$\Rightarrow$ lapse given by $N = \dfrac{\chi}{\Psi}$ with $\chi = 1 + \dfrac{G}{2c^2}\left(\dfrac{\beta_1}{r_1} + \dfrac{\beta_2}{r_2}\right)$

Skeleton field for 2 black holes (II)

Shift function

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Shift function

- 2nd skeleton approximation: 2PN flesh term neglected in N^i

$$\partial_j \left(\Psi^{-6} N \pi^{i}{}_{j} \right) \to \Psi^{-6} N \partial_j \pi^{i}{}_{j} = -\frac{8\pi G}{c^3} \Psi^{-6} N \sum_{A=1}^{N} p_{Ai} \delta_A$$

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- integration of the shift equation $\sim$ integration of $\mathcal{J}_i$

$$N^i = \frac{G}{c^3} \sum_{A=1}^{N} \chi_A \Psi_A^{-7} \left(\frac{1}{2} p_{Aj} \partial_{ij} r_A - 4 p_{Ai} \frac{1}{r_A} \right)$$

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Komar mass M:

$$N=1-\frac{GM}{c^2|\mathbf{x}|}+\mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right)\quad\Rightarrow\quad M=\frac{1}{2}\sum_{A=1}^{2}(\alpha_A+\beta_A)$$

Equations of motion

Obtained by means of an implicit differentiation

$$\dot{x}_1^i = \frac{c^2}{(1-\zeta_1\zeta_2)}[(1-\zeta_2)\theta_{11}^i + (1-\zeta_1)\theta_{21}^i]$$

$$\dot{p}_{1i} = -\frac{c^2}{(1-\zeta_1\zeta_2)}[(1-\zeta_2)\eta_1^i - (1-\zeta_1)\eta_2^i] - \frac{2c^4}{G}(\Psi_1 + \Psi_2 - 2)n_{12}^i$$

with

$$\tilde{\eta}_{A1}^i \equiv -\frac{Gm_A n_{12}^i}{2c^2 r_{12}^2 \Psi_A}\left(1 + \frac{p_A^2}{m_A^2 c^2 \Psi_A^4}\right)^{\frac{1}{2}} + \frac{G\partial_{1i}(p_{Aj}V_{Aj}/r_{12})}{2c^3\Psi_A^7}$$

$$\tilde{\eta}_{A2}^i \equiv -\tilde{\eta}_{A1}^i$$

$$\zeta_A \equiv \frac{\chi_B - 1}{\chi_A}$$

$$\tilde{\theta}_{A1}^i \equiv \frac{Gp_{1i}}{2m_A c^4 r_{12}\Psi_A^5}\left(1 + \frac{p_A^2}{m_A^2 c^2 \Psi_A^4}\right)^{-\frac{1}{2}} + \frac{G\,\partial_{p_{1i}}(p_{Aj}V_{Aj})}{2c^3 r_{12}\Psi_A^7}$$

CIRCULAR MOTION

Circular orbits

PN calculation of $\hat{H} = \{H - (m_1 + m_2)c^2\}/\mu$, [$\mu$ = reduced mass]

$$\hat{H} = \sum_{A=1}^{2} \sum_{i=1}^{n+1} \frac{\alpha_A^{(i)}}{\mu} \varepsilon^i + \mathcal{O}\left(\varepsilon^{n+2}\right)$$

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Circularity:

- $n_{12}^i p_{1i} = n_{12}^i p_{2i} = 0,$
- $p_{1i} p_{1i} = \dfrac{G(m_1 + m_2)}{r_{12}^2} \mu^2 j^2$
- $\dfrac{\partial \hat{H}(r_{12}, j)}{\partial r_{12}} = 0$

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theorem: ADM mass = Komar mass

Equality of Komar and ADM mass (I)

- dependence of α_A for circular motion

$$\boldsymbol{\alpha_A = \alpha_A(\Psi_A(r,j), r, j)}$$

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- relation between α_A and β_A

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- implication of circularity

$$\frac{\partial H(r_{12}, j)}{\partial r_{12}} = 0 \quad \Longleftrightarrow \quad \sum_{A=1}^{2} \frac{\partial \alpha_A(r_{12}, j)}{\partial r_{12}} = 0$$

$$\Longleftrightarrow \quad \sum_{A=1}^{2} \left(-\frac{\beta_A}{\chi_A}\frac{\partial \Psi_A}{\partial r_{12}} + \frac{1}{2r_{12}}\left(\alpha_A - \beta_A \frac{\Psi_A}{\chi_A}\right)\right) = 0$$

Equality of Komar and ADM mass (II)

- expression of $\partial\Psi_A/\partial r_{12}$

$$\Psi_A(r,j) = 1 + \frac{G\alpha_B}{2r_{12}c^2}$$
$$\Rightarrow \frac{\partial\Psi_A}{\partial r_{12}} = -\frac{G}{4r_{12}c^2}\left(\alpha_A + \beta_A\frac{\Psi_B}{\chi_B}\right) + \frac{\beta_B}{\chi_B}\frac{\partial\Psi_B}{\partial r_{12}}$$

$\hookleftarrow$ 2 equations for 2 unknowns

solution $\dfrac{\partial\Psi_A}{\partial r_{12}} = -\dfrac{G}{4r_{12}c^2}(\alpha_B + \beta_B)$, $B \neq A$

- insertion in the circularity condition $\partial H(r_{12},j)/\partial r_{12} = 0$

result $$\sum_{A=1}^{2}(\alpha_A - \beta_A) = 0 \quad \Longleftrightarrow \quad M_{\text{ADM}} = M_{\text{Komar}}$$

Application: last stable circular orbit