Numerical approaches to the relativistic two-body problem:

constructing initial data

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Plan

1. 3+1 formalism of general relativity
2. Solving the constraint equations
   (a) Conformal transverse traceless method
   (b) Conformal thin sandwich method
3. Compact binaries in circular orbits
   (a) Effective potential approach
   (b) Helical Killing vector approach

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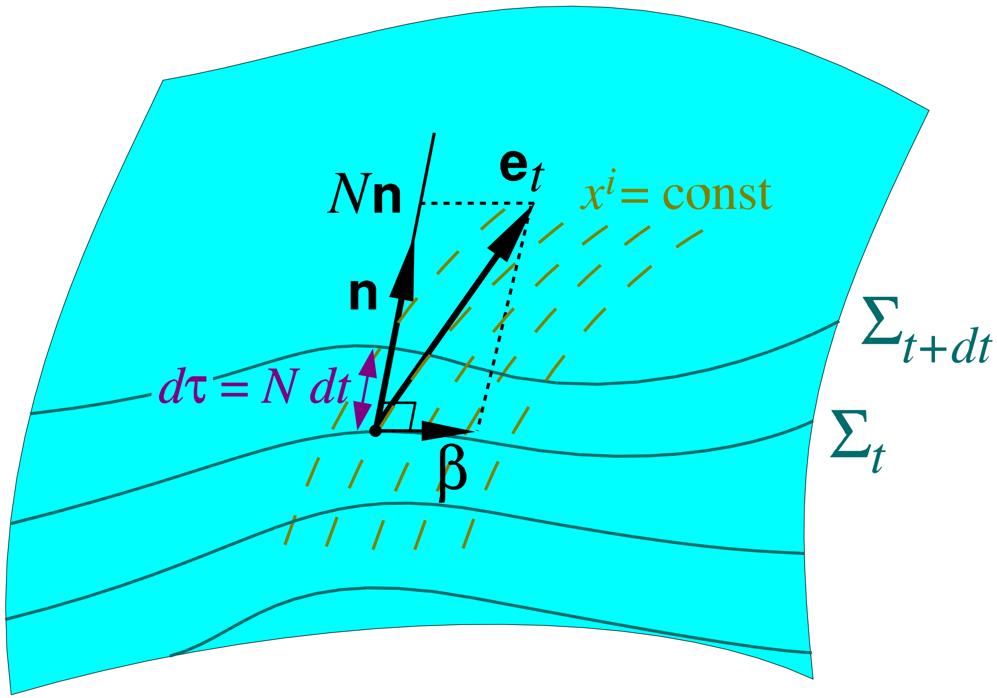
The 3+1 formalism of general relativity

3+1 formalism

History: Lichnerowicz (1944), Choquet-Bruhat (1952), Arnowitt, Deser & Misner (1962), York & Ó Murchadha (1974), and many others...

Basics: Foliation of spacetime by a family of spacelike hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$; on each hypersurface, pick a coordinate system $(x^i)_{i\in\{1,2,3\}}$
$\Longrightarrow (x^\mu)_{\mu\in\{0,1,2,3\}} = (t, x^1, x^2, x^3)$ = coordinate system on spacetime (t = time coordinate, without any particular physical significance)



$\mathbf{n}$: future directed unit normal to Σ_t :
$\mathbf{n} = -N\,\mathbf{d}t$, N : lapse function
$\mathbf{e}_t = \partial/\partial t$: time vector of the natural basis associated with the coordinates (x^μ)

$$\left.\begin{array}{l} N \ : \ \text{lapse function} \\ \boldsymbol{\beta} \ : \ \text{shift vector} \end{array}\right\} \ \mathbf{e}_t = N\mathbf{n} + \boldsymbol{\beta}$$

Geometry of the hypersurfaces Σ_t:
– induced metric $\boldsymbol{\gamma} = \mathbf{g} + \mathbf{n} \otimes \mathbf{n}$
– extrinsic curvature : $\mathbf{K} = -\frac{1}{2}\pounds_{\mathbf{n}}\boldsymbol{\gamma}$

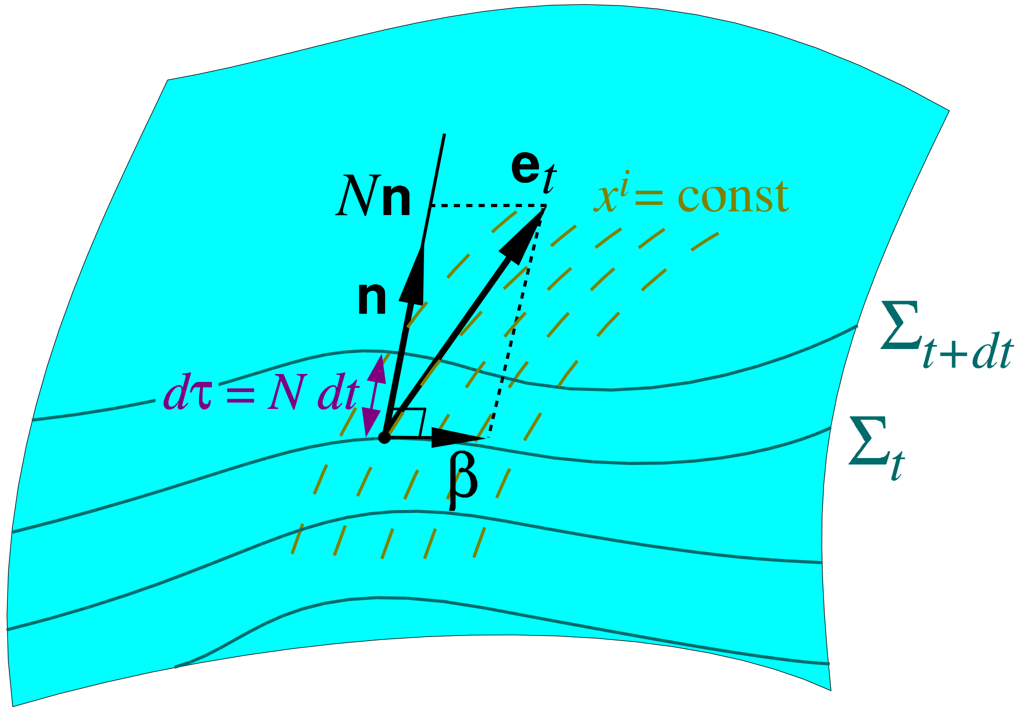
$$\boxed{g_{\mu\nu}\,dx^\mu\,dx^\nu = -N^2\,dt^2 + \gamma_{ij}\,(dx^i + \beta^i dt)\,(dx^j + \beta^j dt)}$$

Choice of coordinates and 3+1 formalism

$$(x^\mu) = (t, x^i) = (t, x^1, x^2, x^3)$$

Choice of lapse function N $\iff$ choice of the slicing (Σ_t)

Choice of shift vector $\boldsymbol{\beta}$ $\iff$ choice of spatial coordinates (x^i) in each hypersurface Σ_t (via the choice of $\mathbf{e}_t$)



A widely chosen foliation : maximal slicing : $K := \mathrm{tr}\, \mathbf{K} = 0$

3+1 decomposition of Einstein equation

Orthogonal projection of Einstein equation onto Σ_t and along the normal to Σ_t :

- Hamiltonian constraint: $\boxed{R + K^2 - K_{ij}K^{ij} = 16\pi E}$
- Momentum constraint : $\boxed{D_j K^{ij} - D^i K = 8\pi J^i}$
- Dynamical equations :

$$\boxed{\frac{\partial K_{ij}}{\partial t} - \pounds_{\boldsymbol{\beta}} K_{ij} = -D_i D_j N + N\left[R_{ij} - 2K_{ik}K^k_{\ j} + KK_{ij} + 4\pi((S-E)\gamma_{ij} - 2S_{ij})\right]}$$

$E := \mathbf{T}(\mathbf{n},\mathbf{n}) = T_{\mu\nu}\, n^\mu n^\nu$, $\quad J_i := -\gamma_i^{\ \mu}\, T_{\mu\nu}\, n^\nu$, $\quad S_{ij} := \gamma_i^{\ \mu}\, \gamma_j^{\ \nu}\, T_{\mu\nu}$, $\quad S := S_i^{\ i}$

D_i : covariant derivative associated with $\boldsymbol{\gamma}$, $\quad R_{ij}$: Ricci tensor of D_i, $\quad R := R_i^{\ i}$

Kinematical relation between $\boldsymbol{\gamma}$ and $\mathbf{K}$: $\quad \dfrac{\partial \gamma^{ij}}{\partial t} + D^i\beta^j + D^j\beta^i = 2NK^{ij}$

Formal. 3+1 $\Longrightarrow$ Resolution of Einstein equation $\equiv$ Cauchy problem [Choquet-Bruhat 1952]

Conformal metric

York (1972) : **Dynamical degrees of freedom** of the gravitational field carried by the conformal "metric"

$$\hat{\gamma}_{ij} := \gamma^{-1/3} \gamma_{ij} \qquad \text{with } \gamma := \det \gamma_{ij}$$

$$\hat{\gamma}_{ij} = \text{tensor density of weight } -2/3$$

To work with tensor fields only, introduce an *extra structure* on Σ_t: a flat metric **f** such that $\dfrac{\partial f_{ij}}{\partial t} = 0$ and $\gamma_{ij} \sim f_{ij}$ at spatial infinity (asymptotic flatness)

Define $\boxed{\tilde{\gamma}_{ij} := \Psi^{-4} \gamma_{ij}}$ or $\boxed{\gamma_{ij} =: \Psi^4 \tilde{\gamma}_{ij}}$ with $\boxed{\Psi := \left(\frac{\gamma}{f}\right)^{1/12}}$, $f := \det f_{ij}$

$\tilde{\gamma}_{ij}$ is invariant under any conformal transformation of γ_{ij} and verifies $\det \tilde{\gamma}_{ij} = f$

Notations: $\tilde{\gamma}^{ij}$: inverse conformal metric : $\tilde{\gamma}_{ik} \tilde{\gamma}^{kj} = \delta_i^{\ j}$
$\tilde{D}_i$: covariant derivative associated with $\tilde{\gamma}_{ij}$, $\tilde{D}^i := \tilde{\gamma}^{ij} \tilde{D}_j$
$\mathcal{D}_i$: covariant derivative associated with f_{ij}, $\mathcal{D}^i := f^{ij} \mathcal{D}_j$

Conformal decomposition

Relation between the Ricci tensor **R** of γ at the Ricci tensor $\tilde{\mathbf{R}}$ of $\tilde{\gamma}$:

$$R_{ij} = \tilde{R}_{ij} - 2\tilde{D}_i\tilde{D}_j \ln\Psi + 4\tilde{D}_i \ln\Psi \, \tilde{D}_j \ln\Psi - 2\left(\tilde{D}^k\tilde{D}_k \ln\Psi + 2\tilde{D}_k \ln\Psi \, \tilde{D}^k \ln\Psi\right) \tilde{\gamma}_{ij}$$

Trace : $R = \Psi^{-4}\left(\tilde{R} - 8\tilde{D}_k\tilde{D}^k \ln\Psi - 8\tilde{D}_k \ln\Psi \, \tilde{D}^k \ln\Psi\right)$

Conformal representation of the traceless part of the extrinsic curvature:

$$A^{ij} := \Psi^4\left(K^{ij} - \frac{1}{3}K\gamma^{ij}\right)$$

Indices lowered with the conformal metric: $A_{ij} := \tilde{\gamma}_{ik}\tilde{\gamma}_{jl}A^{kl} = \Psi^{-4}\left(K_{ij} - \frac{1}{3}K\gamma_{ij}\right)$

Conformal decomposition of Einstein equations

Hamiltonian constraint $\longrightarrow$ $\tilde{D}_i\tilde{D}^i\Psi = \frac{\Psi}{8}\tilde{R} - \Psi^5\left(2\pi E + \frac{1}{8}A_{ij}A^{ij} - \frac{K^2}{12}\right)$

Momentum constraint $\longrightarrow$ $\tilde{D}_j A^{ij} + 6A^{ij}\tilde{D}_j \ln\Psi - \frac{2}{3}\tilde{D}^i K = 8\pi\Psi^4 J^i$

Trace of the evolution equation for **K** $\longrightarrow$

$$\frac{\partial K}{\partial t} - \beta^i\tilde{D}_i K = -\Psi^{-4}\left(\tilde{D}_i\tilde{D}^i N + 2\tilde{D}_i\ln\Psi\,\tilde{D}^i N\right) + N\left[4\pi(E+S) + A_{ij}A^{ij} + \frac{K^2}{3}\right],$$

combined with the Hamiltonian constr. $\longrightarrow$ equation for $\boxed{Q := \Psi^2 N}$:

$$\begin{aligned}\tilde{D}_i\tilde{D}^i Q \quad &= \quad \Psi^6\left[N\left(4\pi S + \frac{3}{4}A_{ij}A^{ij} + \frac{K^2}{2}\right) - \frac{\partial K}{\partial t} + \beta^i\tilde{D}_i K\right] \\ &\quad + \Psi^2\left[N\left(\frac{1}{4}\tilde{R} + 2\tilde{D}_i\ln\Psi\,\tilde{D}^i\ln\Psi\right) + 2\tilde{D}_i\ln\Psi\,\tilde{D}^i N\right]\end{aligned}$$

Conformal decomposition of Einstein equations (con't)

Traceless part of the evolution equation for **K** $\longrightarrow$

$$\begin{aligned}
\frac{\partial A^{ij}}{\partial t} \quad - \quad & \pounds_{\boldsymbol{\beta}} A^{ij} - \frac{2}{3}\tilde{D}_k\beta^k\, A^{ij} = -\Psi^{-6}\left(\tilde{D}^i\tilde{D}^jQ - \frac{1}{3}\tilde{D}_k\tilde{D}^kQ\,\tilde{\gamma}^{ij}\right) \\
& +\Psi^{-4}\left\{N\left(\tilde{\gamma}^{ik}\tilde{\gamma}^{jl}\tilde{R}_{kl} + 8\tilde{D}^i\ln\Psi\,\tilde{D}^j\ln\Psi\right) + 4\left(\tilde{D}^i\ln\Psi\,\tilde{D}^jN + \tilde{D}^j\ln\Psi\,\tilde{D}^iN\right)\right. \\
& \left. -\frac{1}{3}\left[N\left(\tilde{R} + 8\tilde{D}_k\ln\Psi\tilde{D}^k\ln\Psi\right) + 8\tilde{D}_k\ln\Psi\tilde{D}^kN\right]\,\tilde{\gamma}^{ij}\right\} \\
& +N\left[KA^{ij} + 2\tilde{\gamma}_{kl}A^{ik}A^{jl} - 8\pi\left(\Psi^4S^{ij} - \frac{1}{3}S\tilde{\gamma}^{ij}\right)\right]
\end{aligned}$$

Conformal decomposition of the kinematical relation between γ and K

Relation between the extrinsic curvature and the time derivative of the metric:

$$\frac{\partial \gamma^{ij}}{\partial t} + D^i \beta^j + D^j \beta^i = 2NK^{ij}$$

- trace part $\longrightarrow$ $$\frac{\partial \Psi}{\partial t} = \beta^i \tilde{D}_i \Psi + \frac{\Psi}{6}\left(\tilde{D}_i \beta^i - NK\right)$$
- traceless part $\longrightarrow$ $$\frac{\partial \tilde{\gamma}^{ij}}{\partial t} = 2NA^{ij} - (\tilde{L}\beta)^{ij}$$

with the conformal Killing operator acting on the shift vector being defined as

$$\boxed{(\tilde{L}\beta)^{ij} := \tilde{D}^j \beta^i + \tilde{D}^i \beta^j - \frac{2}{3}\tilde{D}_k \beta^k \, \tilde{\gamma}^{ij}}$$

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Solving the constraint equations

General remarks

Solving the constraint equations $\Longrightarrow$ get **initial data** $(\boldsymbol{\gamma}, \mathbf{K})$ for the Cauchy problem of the 3+1 formalism

- **Hamiltonian constraint:** quasilinear elliptic equation for the conformal factor Ψ
- **Momentum constraint:** fix the divergence of A^{ij} (with respect to $\tilde{D}$)

Basic property: the constraint equations are preserved by the evolution equations
Consequently one may choose between

- a free evolution schemes (constraint equations used only to check the numerical solution)
- a constrained evolution schemes (solve the constraint equations at each step)

cf. T. Baumgarte's talk

Methods to solve the constraint equations

- Conformal transverse-traceless method (York & Ó Murchadha) [this talk]
- Conformal thin sandwich (York) [this talk]
- Gluing techniques (Isenberg, Mazzeo, Pollack, Corvino, Schoen)
- Quasi-spherical (Bartnik, Sharples)

2.1

The conformal transverse-traceless method

The conformal transverse-traceless (CTT) method

Origin: York (1979), variant of Ó Murchadha & York (1974)

Split K^{ij} into a traceless part $K^{ij}_{\rm T}$ and a trace part : $K^{ij} = K^{ij}_{\rm T} + \frac{K}{3}\gamma^{ij}$

Motivated by the identity $D_j K^{ij}_{\rm T} = \Psi^{-10}\tilde D_j(\Psi^{10} K^{ij}_{\rm T})$,
introduce a conformal traceless extrinsic curvature $\tilde A^{ij}$ by $K^{ij}_{\rm T} =: \Psi^{-10}\tilde A^{ij}$
NB: $\tilde A^{ij} = \Psi^6 A^{ij}$

Split $\tilde A^{ij}$ into a longitudinal and transverse part: $\boxed{\tilde A^{ij} = (\tilde L X)^{ij} + \tilde A^{ij}_{\rm TT}}$

with $\quad (\tilde L X)^{ij} := \tilde D^j X^i + \tilde D^i X^j - \frac{2}{3}\tilde D_k X^k\,\tilde\gamma^{ij}$ (conformal Killing operator)

and $\quad \tilde D_j \tilde A^{ij}_{\rm TT} = 0$ (transversality with respect to $\tilde\gamma$)

Finally: $\boxed{K^{ij} = \Psi^{-10}\left[(\tilde L X)^{ij} + \tilde A^{ij}_{\rm TT}\right] + \frac{K}{3}\gamma^{ij}}$

Constraint equations in the CTT framework

Hamiltonian constraint $\searrow$ (Lichnerowicz equation)

$$\boxed{\tilde{D}_i \tilde{D}^i \Psi = \frac{\Psi}{8} \tilde{R} - \Psi^5 \left(2\pi E - \frac{K^2}{12} \right) - \frac{1}{8} \tilde{A}_{ij} \tilde{A}^{ij} \, \Psi^{-7}} \qquad (1)$$

Momentum constraint $\searrow$

$$\boxed{\tilde{D}_k \tilde{D}^k X^i + \frac{1}{3} \tilde{D}^i \tilde{D}_k X^k + \tilde{R}^i{}_j X^j = 8\pi \Psi^{10} J^i + \frac{2}{3} \Psi^6 \tilde{D}^i K} \qquad (2)$$

Freely specifiable data: $(\tilde{\gamma}_{ij}, K, \tilde{A}^{ij}_{\mathrm{TT}})$ and (E, J^i), with

- $\tilde{\gamma}_{ij}$ symmetric, positive definite
- $\tilde{A}^{ij}_{\mathrm{TT}}$ symmetric, transverse and traceless with respect to $\tilde{\gamma}_{ij}$

Procedure: solve (1) and (2) to get Ψ and X^i; the valid initial data is then

$$\gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij} \quad \text{and} \quad K^{ij} = \Psi^{-10} \left[(\tilde{L}X)^{ij} + \tilde{A}^{ij}_{\mathrm{TT}} \right] + \frac{K}{3} \gamma^{ij}$$

Remarks about the CTT constraint equations

- The Hamiltonian constraint (1) is a quasilinear elliptic equation for Ψ
- The momentum constraint (2) is a linear vector elliptic equation for X^i
- If one chooses **maximal slicing**, $K = 0$ and (2) becomes independent from Ψ :

$$\tilde{D}_k \tilde{D}^k X^i + \frac{1}{3} \tilde{D}^i \tilde{D}_k X^k + \tilde{R}^i{}_j X^j = 8\pi \tilde{J}^i$$

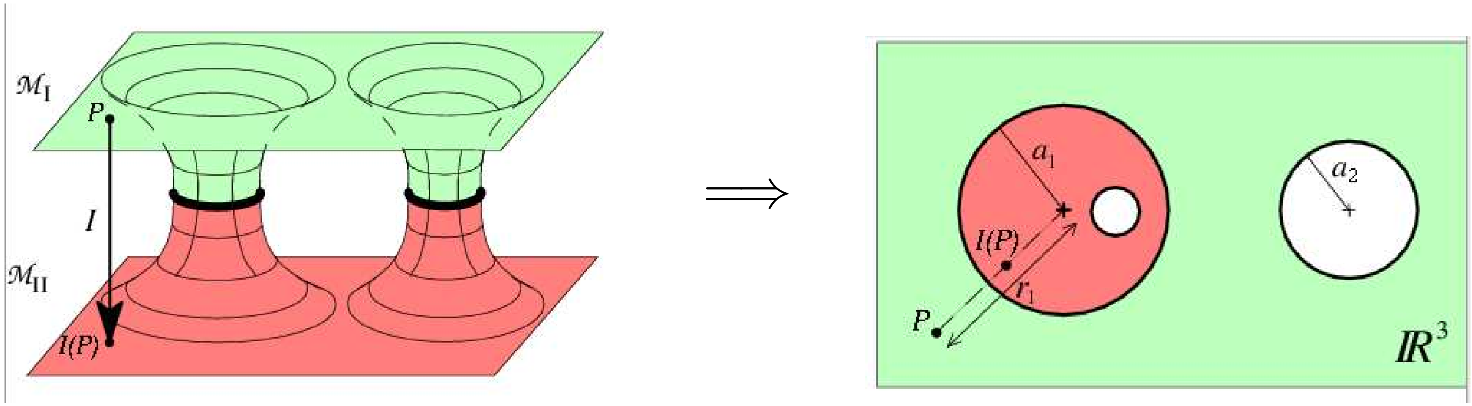
(provided one selects $\tilde{J}^i := \Psi^{10} J^i$ as the matter freely specifiable data)

Boundary conditions

Topology of the initial data manifold Σ_0 :

- **for neutron star spacetimes:** $\Sigma_0 \sim \mathbb{R}^3$
- **for black hole spacetimes:** $\Sigma_0 \sim \mathbb{R}^3 \setminus$ some balls (half of Misner-Lindquist topology) or $\Sigma_0 \sim \mathbb{R}^3 \setminus$ some points *(punctures)* (Brill-Linquist topology)

Example: Misner-Lindquist topology for two black holes:



Constraint equations (1) and (2) = *elliptic* equations $\Longrightarrow$ **boundaries conditions** have to be supplied at the inner boundaries and outer boundary (spatial infinity) of Σ_0 to yield a unique solution

At spatial infinity : $\Psi|_{r\to\infty} = 1$ and $X^i|_{r\to\infty} = 0$
(asymptotic flatness for $\tilde{\gamma}_{ij} \underset{r\to\infty}{\sim} f_{ij}$)

At some inner sphere $\mathcal{S}$: for example, Ψ such that $\mathcal{S}$ = apparent horizon

Global quantities as surface integrals at spatial infinity

Asymptotic flatness for $r \to \infty$ (Cartesian components):

- $\gamma_{ij} = f_{ij} + O(r^{-1}) \iff \Psi = 1$ and $\tilde\gamma_{ij} = f_{ij} + O(r^{-1})$ (NB: $f^{ij}\tilde\gamma_{ij} = 1 + O(r^{-2})$)
- $\mathcal{D}_k \gamma_{ij} = O(r^{-2}) \iff \mathcal{D}_k \Psi = O(r^{-2})$ and $\mathcal{D}_k \tilde\gamma_{ij} = O(r^{-2})$ (no grav. wave at spatial inf.)
- $K^{ij} = O(r^{-2})$

∘ quasi-isotropic gauge : additional condition: $\mathcal{D}^j \tilde\gamma_{ij} = O(r^{-3})$ [York 1979]

- **ADM mass :** $M_{\rm ADM} = \dfrac{1}{16\pi} \oint_\infty \left(\mathcal{D}^j \gamma_{ij} - f^{jk} \mathcal{D}_i \gamma_{jk} \right) dS^i$

  ⋆ in the quasi-isotropic gauge: $M_{\rm ADM} = -\dfrac{1}{2\pi} \oint_\infty \mathcal{D}_i \Psi \, dS^i$ (function of Ψ only)

- **ADM linear momentum :** $P^i_{\rm ADM}$, projections along three independent translational Killing vectors of **f**, $\boldsymbol{\xi}_{(i)}$:

$$P_{j\,{\rm ADM}}\, \xi^j_{(i)} = \frac{1}{8\pi} \oint_\infty \left(K_{jk} - K f_{jk} \right) \xi^j_{(i)} \, dS^k$$

- **Angular momentum : defined only within the quasi-isotropic gauge** : projections along three independent rotational Killing vectors of **f**, $\boldsymbol{\eta}_{(i)}$:

$$J_j\, \xi^j_{(i)} = \frac{1}{8\pi} \oint_\infty \left(K_{jk} - K f_{jk} \right) \eta^j_{(i)} \, dS^k$$

Conformally flat initial data

As a part of the freely specifiable data, choose $\boxed{\tilde{\gamma}_{ij} = f_{ij}}$ (flat metric)

Consequently $\tilde{D}_i = \mathcal{D}_i$ and $\tilde{R}_{ij} = 0$

Choose also $\boxed{K = 0}$ (maximal slicing)

Then the Hamiltonian constraint (1) becomes

$$\Delta\Psi = -2\pi\Psi^5 E - \frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij}\,\Psi^{-7}$$

and the momentum constraint (2) reduces to

$$\Delta X^i + \frac{1}{3}\mathcal{D}^i\mathcal{D}_k X^k = 8\pi\tilde{J}^i$$

where $\Delta := f^{ij}\mathcal{D}_i\mathcal{D}_j$ is the flat space Laplacian

The Bowen-York solution

In addition to $\tilde{\gamma}_{ij} = f_{ij}$ and $K = 0$, choose $E = 0$ and $J^i = 0$ (vacuum spacetime), as well as $\tilde{A}^{ij}_{\mathrm{TT}} = 0$.

Then

$$\text{Hamiltonian constraint} \Rightarrow \quad \Delta\Psi = -\frac{\Psi^{-7}}{8}\tilde{A}_{ij}\tilde{A}^{ij} \tag{3}$$

$$\text{Momentum constraint} \Rightarrow \quad \Delta X^i + \frac{1}{3}\mathcal{D}^i\mathcal{D}_k X^k = 0 \tag{4}$$

Bowen-York analytical solution of (4) [Bowen & York, PRD **21**, 2047 (1980)] :

For a single black hole : $X^i_{\mathrm{BY}_0} = -\frac{1}{4r}\left(7P^i + P_j\frac{x^j x^i}{r^2}\right) - \frac{1}{r^3}\epsilon^i{}_{jk}S^j x^k$

with $x^i = (x, y, z)$, $r^2 := x^2 + y^2 + z^2$

Two constant vector parameters : $\begin{cases} P^i & = \text{ADM linear momentum} \\ S^i & = \text{angular momentum} \end{cases}$

The Bowen-York solution (con't)

Example: choose S^i perpendicular to P^i and choose Cartesian coordinate system (x, y, z) such that $P^i = (0, P, 0)$ and $S^i = (0, 0, S)$. Then

$$
\begin{aligned}
X^x_{\mathrm{BY}_0} &= -\frac{P}{4}\frac{xy}{r^3} + S\frac{y}{r^3} \\
X^y_{\mathrm{BY}_0} &= -\frac{P}{4r}\left(7 + \frac{y^2}{r^2}\right) - S\frac{x}{r^3} \\
X^z_{\mathrm{BY}_0} &= -\frac{P}{4}\frac{xz}{r^3}
\end{aligned}
$$

Bowen-Tork extrinsic curvature: $\tilde{A}^{ij}_{\mathrm{BY}_0} = (\bar{L}X_{\mathrm{BY}_0})^{ij}$

$$
\boxed{\tilde{A}^{ij}_{\mathrm{BY}_0} = \frac{3}{2r^3}\left[P^i x^j + P^j x^i - \left(\delta^{ij} - \frac{x^i x^j}{r^2}\right) P^k x_k\right] + \frac{3}{r^5}\left(\epsilon^i{}_{kl} S^k x^l x^j + \epsilon^j{}_{kl} S^k x^l x^i\right)}
$$

There remains to solve (numerically) the non-linear elliptic equation (3) to get Ψ.

Static Bowen-York solution = Schwarzschild solution

Static case: $P^i = 0$ and $S^i = 0$

$\Longrightarrow X^i = 0$ and $\tilde{A}^{ij} = 0$

Hamiltonian constraint (3) $\rightarrow$ $\quad \Delta\Psi = 0$

Non trivial spherically symmetric solution : $\Psi = 1 + \dfrac{M}{2r}$

Hence one recovers **Schwarzschild solution in isotropic coordinates**:

$$\gamma_{ij} = \left(1 + \frac{M}{2r}\right)^4 f_{ij}$$

Non-conformally flat initial data

There does not exist any conformally flat axisymmetric slice of Kerr spacetime [Garat & Price, PRD **61**, 124011 (2000)]

Non flat conformal metric: Matzner, Huq & Shoemaker (1998) [PRD **59**, 024015], Marronetti & Matzner (2000) [PRL **85**, 5500] :
linear combination of Kerr-shild metrics:

$$\tilde{\boldsymbol{\gamma}} = \mathbf{f} + 2B_1 H_1 \,\boldsymbol{\ell}_1 \otimes \boldsymbol{\ell}_1 + 2B_2 H_2 \,\boldsymbol{\ell}_2 \otimes \boldsymbol{\ell}_2$$

with $\boldsymbol{\ell}_i$: null vector of a single Kerr-Schild metric

$H_i = \dfrac{M_i r_i}{r_i^2 + a_i^2 \cos^2\theta_i}$

B_i : attenuation functions

2.2

The conformal thin sandwich method

The conformal thin sandwich (CTS) method

Origin: York (1999) [PRL **82**, 1350], Pfeiffer & York (2003), [PRD **67**, 044022]

Use the same conformal decomposition of the extrinsic curvature as in the 3+1 evolution equations:

$$K^{ij} = \Psi^{-4} A^{ij} + \frac{1}{3} K \gamma^{ij}$$

and rewrite the traceless kinematical relation between $\boldsymbol{\gamma}$ and **K** as

$$\boxed{A^{ij} = \frac{1}{2N} \left[(\tilde{L}\beta)^{ij} + \tilde{u}^{ij} \right]}$$

with $\tilde{u}^{ij} := \frac{\partial \tilde{\gamma}^{ij}}{\partial t}$

$\tilde{u}^{ij}$ = **freely specifiable data** (conformal thin sandwich), instead of $\tilde{A}^{ij}_{\mathrm{TT}}$ in the CTT formulation.

Equations in the CTS framework

Hamiltonian constraint $\searrow$

$$\tilde{D}_i\tilde{D}^i\Psi = \frac{\Psi}{8}\tilde{R} - \Psi^5\left(2\pi E + \frac{1}{8}A_{ij}A^{ij} - \frac{K^2}{12}\right) \quad (5)$$

Momentum constraint $\searrow$

$$\tilde{D}_k\tilde{D}^k\beta^i + \frac{1}{3}\tilde{D}^i\tilde{D}_k\beta^k + \tilde{R}^i{}_j\beta^j - (\tilde{L}\beta)^{ij}\tilde{D}_j\ln(N\Psi^{-6}) \; = \; 2N\left(8\pi\Psi^4 J^i + \frac{2}{3}\tilde{D}^i K\right) - \tilde{D}_j\tilde{u}^{ij} + \tilde{u}^{ij}\tilde{D}_j\ln(N\Psi^{-6}) \quad (6)$$

Trace of the evolution equation for **K** $\searrow$ $\quad (\dot{K} := \partial K/\partial t)$

$$\tilde{D}_i\tilde{D}^i N + 2\tilde{D}_i\ln\Psi\,\tilde{D}^i N = \Psi^4\left\{N\left[4\pi(E+S) + A_{ij}A^{ij} + \frac{K^2}{3}\right] + \beta^i\tilde{D}_i K - \dot{K}\right\} \quad (7)$$

Freely specifiable data: $(\tilde{\gamma}_{ij}, \tilde{u}^{ij} = \dot{\tilde{\gamma}}^{ij}, K, \dot{K})$ and (E, J^i)

Equations in the CTS framework (con't)

Freely specifiable data: $(\tilde{\gamma}_{ij}, \tilde{u}^{ij} = \dot{\tilde{\gamma}}^{ij}, K, \dot{K})$ and (E, J^i) with

- $\tilde{\gamma}_{ij}$ symmetric, positive definite
- $\tilde{u}^{ij}$ symmetric and traceless with respect to $\tilde{\gamma}_{ij}$

Procedure: solve (5), (6) and (7) to get Ψ, β^i and N; the valid initial data is then

$$\gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij} \quad \text{and} \quad K^{ij} = \frac{\Psi^{-4}}{2N}\left[(\tilde{L}\beta)^{ij} + \tilde{u}^{ij}\right] + \frac{K}{3}\gamma^{ij}$$

Comparing CTT and CFS

- CTT : choose some transverse traceless part $\tilde{A}^{ij}_{\mathrm{TT}}$ of the extrinsic curvature K^{ij}, i.e. some ***momentum*** [1] $\Longrightarrow$ **CTT = Hamiltonian representation**

- CTS : choose some time derivative $\tilde{u}^{ij}$ of the conformal metric $\tilde{\gamma}^{ij}$, i.e. some ***velocity*** $\Longrightarrow$ **CTS = Lagrangian representation**

Advantage of CTT : mathematical theory well developed (at least for constant mean curvature ($K = \mathrm{const}$) slices)

Advantage of CTS : better suited to the description of quasi-stationary spacetimes ($\rightarrow$ quasiequilibrium initial data) :

$$\frac{\partial}{\partial t} \text{ Killing vector} \Rightarrow u^{ij} = 0$$

[1] recall the relation $\pi^{ij} = \sqrt{\gamma}(K\gamma^{ij} - K^{ij})$ between K^{ij} and the ADM canonical momentum

Numerical comparison of CTT and CFS for binary balck holes

[Pfeiffer, Cook & Teukolsky, PRD **66**, 024047 (2002)]

Settings:

- Initial slice $\Sigma_0 = \mathbb{R}^3 \setminus$ two balls
- Choice of freely specifiable pieces:
  - ⋆ $\tilde{\gamma}$ = superposition of two boosted Kerr-Schild metrics
  - ⋆ $K = K_1^{\rm KS} + K_2^{\rm KS}$
  - ⋆ for CTT : $\tilde{A}^{ij}_{\rm TT}$ from a linear superposition of two Kerr-Schild extrinsic curvatures [2]
  - ⋆ for CFS : $\tilde{u}^{ij} = 0$
- Fix the total angular momentum and the proper separation between the two apparent horizons

Results:

- significant differences (5%) in the ADM mass among the two methods
- choice of the freely speciable part of the extrinsic curvature more important than the choice of the conformal metric (even if a flat $\tilde{\gamma}$ is chosen)

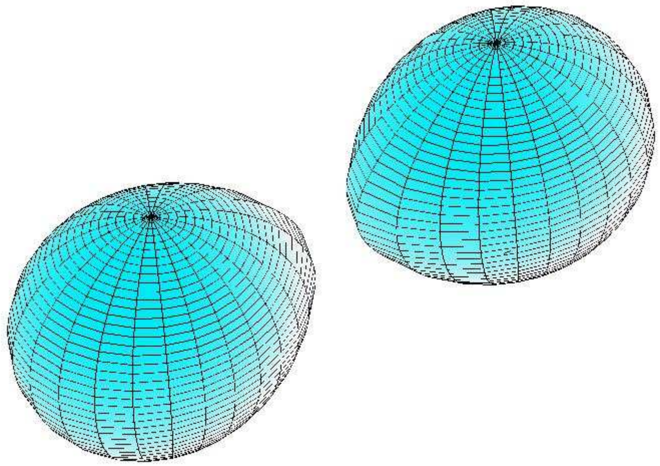
[2] Such computations have also been performed recently by [Bonning et al., gr-qc/0305071]

3

Compact binaries in circular orbits

Astrophysically relevant initial data

Position of the problem: Among all the possible solutions $(\Sigma_0, \gamma, \mathbf{K})$ of the constraint equations, how to pick those which correspond to a binary system in a nearly circular orbit ?



Basically two approaches have been employed in numerical studies:

- **the effective potential approach**, based on CTT [binary black holes]
- **the helical Killing vector approach**, based of CTS [binary black holes, binary neutron stars]

3.1

The Effective Potential approach

The Effective Potential approach (Cook 1994)

Procedure to get a quasiequilibrium configuration of binary black hole in circular orbit:

- Solve only for the vacuum constraint equations on a spacelike **3-dimensional** surface Σ_0 with a non-trivial topology (for instance the Misner-Lindquist topology or the Brill-Lindquist topology)

- Define the binding energy by $E = M_{\mathrm{ADM}} - M_1 - M_2$

- Define a circular orbit as an extremum of E with respect to proper separation l at fixed angular momentum and BH individual mass:

$$\left.\frac{\partial E}{\partial l}\right|_{M_1,M_2,J} = 0$$

- Compute the orbital angular velocity as $\Omega = \left.\frac{\partial E}{\partial J}\right|_{M_1,M_2,l}$

Ambiguities of the effective potential approach

- Contrary to the ADM mass, the individual masses M_1 and M_2 of each black hole are ill-defined quantities in GR.
  Cook ansatz [PRD **50**, 5025 (1994)] : define the individual mass M_i from the apparent horizon area $\mathcal{A}_i$ and individual spin and via the Christodoulou formula:

$$M_i^2 := \frac{\mathcal{A}_i}{16\pi} + \frac{4\pi S_i^2}{\mathcal{A}_i}$$

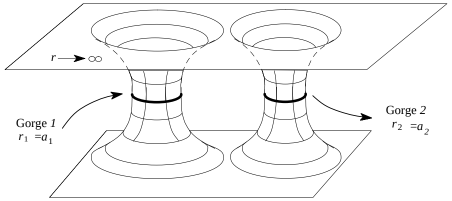
  Caveat 1: Christodoulou formula only established for a single stationary black hole (Kerr spacetime)
  Caveat 2: moreover with $\mathcal{A}_i$ the area of the event horizon, not the apparent one
  Caveat 3: The individual spin S_i suffers from the same lack of unambiguous definition as the individual mass

- No rigorous fundations for the effective potential formulas

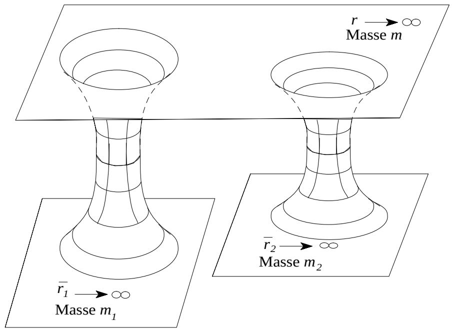
Numerical implementations of the effective potential approach

All based on CTT with (i) conformally flat metric and (ii) Bowen-York extrinsic curvature:

$$K^{ij} = \Psi^{-10} \left[\tilde{A}^{ij}_{\mathrm{BY}_0}(\boldsymbol{P}_1, \boldsymbol{S}_1, x^i \to x^i_1) + \tilde{A}^{ij}_{\mathrm{BY}_0}(\boldsymbol{P}_2, \boldsymbol{S}_2, x^i \to x^i_2) \right]$$



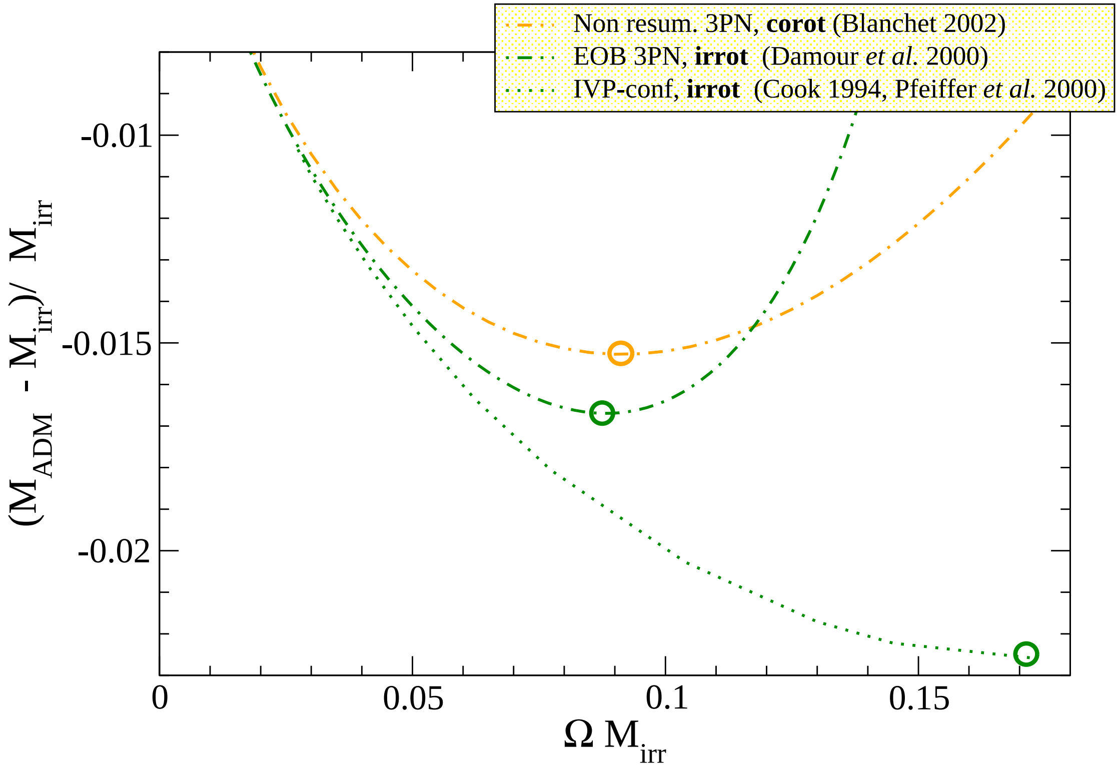
- Cook 1994 [PRD **50**, 5025 (1994)] : *Misner-Lindquist topology*
- Pfeiffer, Teukolsky & Cook 2000 [PRD **62**, 104018 (2000)] : *idem*



- Baumgarte 2000 [PRD **62**, 024018 (2000)] : *Brill-Lindquist topology*

Discrepancy between Effective Potential + Bowen York and post-Newtonian results

Binding energy along an evolutionary sequence of equal-mass binary black holes:



Post-Newtonian computations : at the 3-PN level:

- Damour, Jaranowski & Schäfer 2000 [PRD **62**, 084011 (2000)] : Effective One Body approach (EOB)
- Blanchet 2002 [PRD **65**, 124009 (2002)] : Non-resummed Taylor expansion

3.2

The helical Killing vector approach

Binary systems in quasiequilibrium

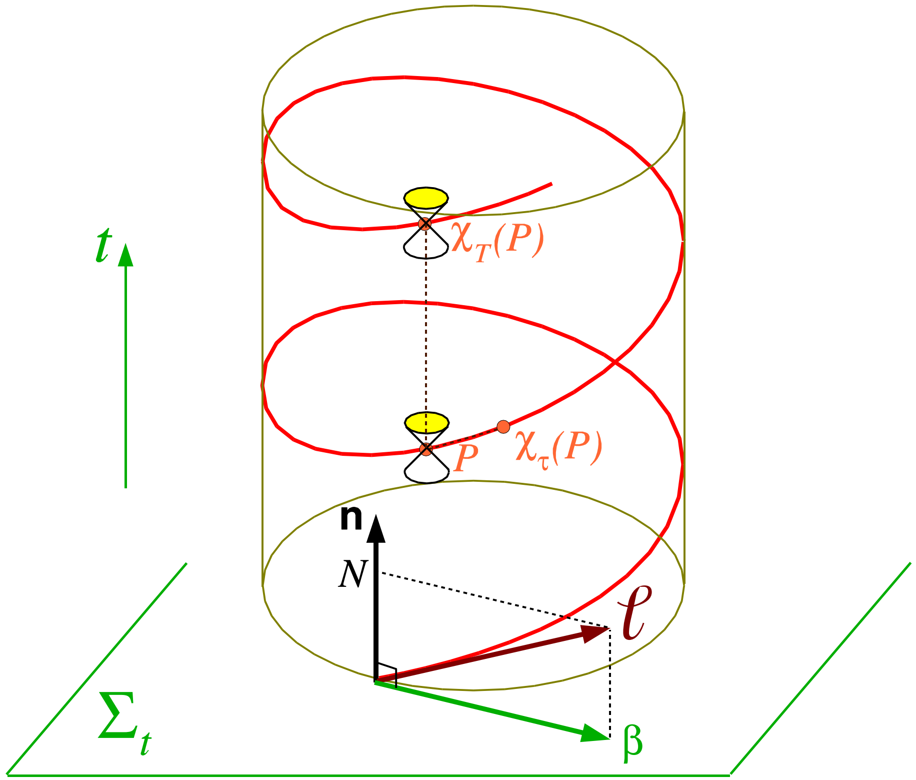
Problem treated: Binary black holes or neutron stars in the pre-coalescence stage
$\Rightarrow$ the notion of **orbit** has still some meaning
Basic idea: Construct an approximate, but full spacetime (i.e. **4-dimensional**) representing 2 orbiting compact objects. Previous numerical treatments: 3-dimensional (initial value problem on a spacelike 3-surface) 4-dimensional approach $\Rightarrow$ rigorous definition of orbital angular velocity

- Binary NS :
  - ⋆ corotating stars : [Baumgarte et al., PRL **79**, 1182 (1997)], [Baumgarte et al., PRD **57**, 7299 (1998)], [Marronetti, Mathews & Wilson, PRD **58**, 107503 (1998)]
  - ⋆ irrotational stars : [Bonazzola, Gourgoulhon & Marck, PRL **82**, 892 (1999)], [Gourgoulhon et al., PRD **63**, 064029 (2001)], [Marronetti, Mathews & Wilson, PRD **60**, 087301 (2000)], [Uryu & Eriguchi, PRD **61**, 124023 (2000)], [Uryu & Eriguchi, PRD **62**, 104015 (2000)], [Taniguchi & Gourgoulhon, PRD **66**, 104019 (2002)], [Taniguchi & Gourgoulhon, submitted (2003)]
  - ⋆ arbitrary spins : [Marronetti & Shapiro, gr-qc/0306075]
- Binary BH :
  - ⋆ corotating BH : [Gourgoulhon, Grandclément & Bonazzola, PRD **65**, 044020 (2002)], [Grandclément, Gourgoulhon & Bonazzola, PRD **65**, 044021 (2002)],
  - ⋆ arbitrary spin : [Cook, PRD **65**, 084003 (2002)]

Helical symmetry

Physical assumption: when the two objects are sufficiently far apart, the radiation reaction can be neglected $\Rightarrow$ closed orbits
Gravitational radiation reaction circularizes the orbits $\Rightarrow$ circular orbits

Geometrical translation: spacetime possesses some **helical symmetry**



Helical Killing vector $\boldsymbol{\ell}$:

(i) timelike near the system,
(ii) spacelike far from it, but such that $\exists$ a smaller $T > 0$ such that the separation between any point P and and its image $\chi_T(P)$ under the symmetry group is timelike

[Bonazzola, Gourgoulhon & Marck, PRD **56**, 7740 (1997)]

[Friedman, Uryu & Shibata, PRD **65**, 064035 (2002)]

Helical symmetry: discussion

Helical symmetry is exact

- in **Newtonian gravity** and in **2nd order Post-Newtonian gravity**
- in general relativity for a non-axisymmetric system (binary) only with **standing gravitational waves**

But a spacetime with a helical Killing vector and standing gravitational waves cannot be asymptotically flat in full GR [Gibbons & Stewart 1983].

We have used a truncated version of GR (the **Isenberg-Wilson-Mathews** approximation, which will be described below) which (i) admits the helical Killing vector and (ii) is asymptotically flat.

Helical symmetry and conformal thin sandwich

Choose coordinates (t, x^i) adapted to the helical Killing vector: $\frac{\partial}{\partial t} = \boldsymbol{\ell}$.

$\Longrightarrow$ the "velocity" part of the freely specifiable data of the CTS approach are fully determined:

$$\boxed{\tilde{u}^{ij} = \frac{\partial \tilde{\gamma}^{ij}}{\partial t} = 0} \quad \text{and} \quad \boxed{\dot{K} = \frac{\partial K}{\partial t} = 0}$$

Remaining free specifiable data: choose

- $\tilde{\gamma}_{ij} = f_{ij}$ (conformal flatness)
- $K = 0$ (maximal slicing)

Helical symmetry and conformal thin sandwich (con't)

CTS equations for $\tilde{\gamma}_{ij} = f_{ij}$ and $K = 0$:

$$\boxed{\Delta\Psi = -\Psi^5 \left(2\pi E + \frac{1}{8} A_{ij} A^{ij}\right)}$$

$$\boxed{\Delta\beta^i + \frac{1}{3}\mathcal{D}^i \mathcal{D}_k \beta^k = 16\pi N \Psi^4 J^i + (\bar{L}\beta)^{ij} \mathcal{D}_j \ln(N\Psi^{-6})}$$

$$\boxed{\Delta N = N\Psi^4 \left[4\pi(E+S) + A_{ij}A^{ij}\right] - 2\mathcal{D}_i \ln\Psi\, \mathcal{D}^i N}$$

where

- $\mathcal{D}_i$ is the covariant derivative associated with the flat metric **f**
- $\Delta := f^{ij}\mathcal{D}_i\mathcal{D}_j$ is the flat Laplacian
- $(\bar{L}\beta)^{ij} := \mathcal{D}^i\beta^j + \mathcal{D}^j\beta^i - \frac{2}{3}\mathcal{D}_k\beta^k\, f^{ij}$
- $A^{ij} = \frac{1}{2N}(\bar{L}\beta)^{ij}$

Helical symmetry and IWM approximation

Isenberg-Wilson-Mathews approximation: waveless approximation to General Relativity based on a conformally flat spatial metric: $\boxed{\boldsymbol{\gamma} = \Psi^4 \boldsymbol{f}}$

[Isenberg (1978)], [Wilson & Mathews (1989)]

$\Rightarrow$ spacetime metric : $ds^2 = -N^2 dt^2 + \Psi^4 f_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$

Amounts to solve only 5 of the 10 Einstein equations:

- Hamiltonian constraint
- momentum constraint (3 equations)
- trace of the evolution equation for the extrinsic curvature

Within the helical symmetry, the IWM equations reduce to the CTS equations

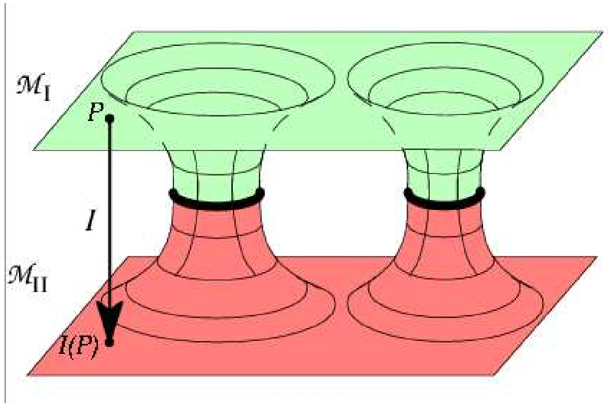
Remaining (non CTS) equation: trace part of the kinematical relation between γ and **K** with $\dfrac{\partial \Psi}{\partial t} = 0$:

$$\mathcal{D}_i \beta^i = -6 \beta^i \mathcal{D}_i \ln \Psi$$

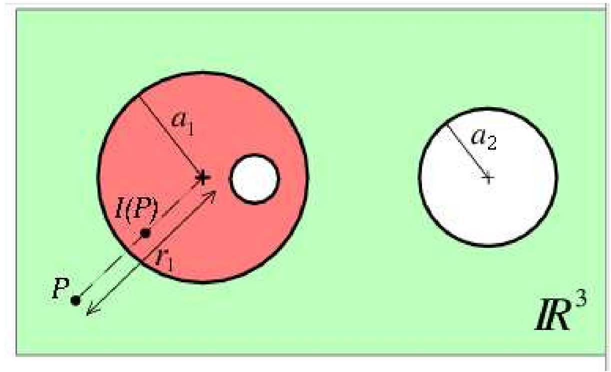
Spacetime manifold

Topology : for binary NS : $\mathbb{R}^4$

for binary BH : $\mathbb{R} \times$ Misner-Lindquist



$\Longrightarrow$



Canonical mapping: $I: \quad (t, r_1, \theta_1, \varphi_1) \mapsto \left(t, \frac{a_1^2}{r_1}, \theta_1, \varphi_1\right)$ isometry

Fluid equation of motion

Neutron star fluid $=$ perfect fluid : $\mathbf{T} = (e+p)\mathbf{u}\otimes\mathbf{u} + p\mathbf{g}$.

Carter-Lichnerowicz equation of motion for zero-temperature fluids:

$$\nabla\cdot\mathbf{T} = 0 \iff \begin{cases} \mathbf{u}\cdot\mathbf{dw} = 0 & (1) \\ \nabla\cdot(n\mathbf{u}) = 0 & (2) \end{cases}$$

$\mathbf{w} := h\mathbf{u}$: co-momentum 1-form
$\mathbf{dw}$: vorticity 2-form

with $n =$ baryon number density and $h = (e+p)/(m_{\rm B}n)$ specific enthalpy.

Cartan identity : Killing vector $\boldsymbol{\ell} \Longrightarrow \pounds_{\boldsymbol{\ell}}\mathbf{w} = 0 = \boldsymbol{\ell}\cdot\mathbf{dw} + \mathbf{d}(\boldsymbol{\ell}\cdot\mathbf{w})$ (3)

Two cases with a first integral : $\boxed{\boldsymbol{\ell}\cdot\mathbf{w} = \mathrm{const}}$ (4)

- **Rigid motion:** $\mathbf{u} = \lambda\boldsymbol{\ell}$: (3) + (1) $\Leftrightarrow$ (4) ; (2) automatically satisfied
- **Irrotational motion:** $\mathbf{dw} = 0 \Leftrightarrow \mathbf{w} = \boldsymbol{\nabla}\Psi$: (3) $\Leftrightarrow$ (4) ; (1) automatically satisfied

$$(2) \Leftrightarrow \frac{n}{h}\nabla\cdot\nabla\Psi + \nabla\left(\frac{n}{h}\right)\cdot\nabla\Psi = 0$$

Astrophysical relevance of the two rotation states

- **Rigid motion (synchronized binaries)** (also called **corotating binaries**) : the viscosity of neutron star matter is far too low to ensure synchronization of the stellar spins with the orbital motion [Kochanek, ApJ **398**, 234 (1992)], [Bildsten & Cutler, ApJ **400**, 175 (1992)] $\Longrightarrow$ **unrealistic state of rotation**

- **Irrotational motion:** good approximation for neutron stars which are not initially millisecond rotators, because then $\Omega_{\rm spin} \ll \Omega_{\rm orb}$ at the late stages.

Rotation state in the binary BH case

Choice: rotation synchronized with the orbital motion (corotating system)

Justifications:

- the only rotation state fully compatible with the helical symmetry
  [Friedman, Uryu & Shibata, PRD **65**, 064035 (2002)]
- for close systems, black hole "effective viscosity" might be very efficient in synchronizing the spins with the orbital motion
  [e.g. Price & Whelan, PRL **87**, 231101 (2001)]

Geometrical translation: the two horizons are **Killing horizons** associated with $\boldsymbol{\ell}$:

$$\boldsymbol{\ell} \cdot \boldsymbol{\ell}|_{\mathcal{H}_1} = 0 \qquad \text{and} \qquad \boldsymbol{\ell} \cdot \boldsymbol{\ell}|_{\mathcal{H}_2} = 0 \; .$$

[cf. the rigidity theorem for a Kerr black hole]

Boundary conditions

Inner boundary (binary BH only):		Spatial infinity:
isometry condition on γ_{rr}: $\left.\left(\frac{\partial\Psi}{\partial r_1}+\frac{\Psi}{2r_1}\right)\right\|_{\mathcal{S}_1}=0$	$\left.\left(\frac{\partial\Psi}{\partial r_2}+\frac{\Psi}{2r_2}\right)\right\|_{\mathcal{S}_2}=0$	asymptotic flatness: $\Psi\to 1$ when $r\to\infty$
corotating black holes: $\boldsymbol{\beta}\|_{\mathcal{S}_1}=0$	$\boldsymbol{\beta}\|_{\mathcal{S}_2}=0$	definition of $\boldsymbol{\ell}$: $\boldsymbol{\beta}\to\Omega\frac{\partial}{\partial\varphi_0}$ when $r\to\infty$
isometry condition on N: $N\|_{\mathcal{S}_1}=0$	$N\|_{\mathcal{S}_2}=0$	asymptotic flatness: $N\to 1$ when $r\to\infty$

Additional equations in the fluid case (binary NS)

Baryon number conservation for irrotational flows:

$$n\underline{\Delta}\Psi + \bar{\nabla}_i n\, \bar{\nabla}^i \Psi = \cdots$$

$\longrightarrow$ singular ($n = 0$ at the stellar surface) elliptic equation to be solved for Ψ.

First integral of fluid motion $\boldsymbol{\ell} \cdot \mathbf{w} = \mathrm{const}$ writes $\quad hN\dfrac{\Gamma}{\Gamma_0} = \mathrm{const} \quad (5)$

with Γ : Lorentz factor between fluid co-moving observer and co-orbiting observer ($= 1$ for synchronized binaries)

Γ_0 : Lorentz factor between co-orbiting observer and asymptotically inertial observer

$\longrightarrow$ solve (5) for the specific enthalpy h.

From h compute the fluid proper energy density e, pressure p and baryon number n via an equation of state:

$$e = e(h), \qquad p = p(h), \qquad n = n(h)$$

Determination of Ω : NS case

First integral of fluid motion:

$$hN\frac{\Gamma}{\Gamma_0} = \mathrm{const}$$

The Lorentz factor Γ_0 contains Ω: at the Newtonian limit, $\ln\Gamma_0$ is nothing but the centrifugal potential: $\ln\Gamma_0 \sim \frac{1}{2}(\mathbf{\Omega}\times\mathbf{r})^2$.

At each step of the iterative procedure, Ω and the location of the rotation axis are then determined so that the stellar centers (density maxima) remain at fixed coordinate distance from each other.

Determination of Ω : BH case

Virial assumption: $O(r^{-1})$ part of the metric $(r \to \infty)$ same as Schwarzschild

[The only quantity "felt" at the $O(r^{-1})$ level by a distant observer is the total mass of the system.]

A priori

$$\Psi \sim 1 + \frac{M_{\rm ADM}}{2r} \qquad \text{and} \qquad N \sim 1 - \frac{M_{\rm K}}{r}$$

Hence

$$\text{(virial assumption)} \iff M_{\rm ADM} = M_{\rm K}$$

Note

$$\text{(virial assumption)} \iff \Psi^2 N \sim 1 + \frac{\alpha}{r^2}$$

Link with the classical virial theorem

Einstein equations $\Rightarrow$

$$\underline{\Delta} \ln(\Psi^2 N) \;=\; \Psi^4 \left[4\pi S_i{}^i + \frac{3}{4}\hat{A}_{ij}\hat{A}^{ij}\right] - \frac{1}{2}\left[\bar{\nabla}_i \ln N \bar{\nabla}^i \ln N + \bar{\nabla}_i \ln(\Psi^2 N)\bar{\nabla}^i \ln(\Psi^2 N)\right]$$

No monopolar $1/r$ term in $\Psi^2 N \iff$

$$\int_{\Sigma_t} \left\{4\pi S_i{}^i + \frac{3}{4}\hat{A}_{ij}\hat{A}^{ij} - \frac{\Psi^{-4}}{2}\left[\bar{\nabla}_i \ln N \bar{\nabla}^i \ln N + \bar{\nabla}_i \ln(\Psi^2 N)\bar{\nabla}^i \ln(\Psi^2 N)\right]\right\} \Psi^4 \sqrt{f}\, d^3x \;=\; 0$$

Newtonian limit is the classical virial theorem:

$$2E_{\rm kin} + 3P + E_{\rm grav} = 0$$

Defining an evolutionary sequence: BH case

An evolutionary sequence is defined by:

$$\left.\frac{dM_{\rm ADM}}{dJ}\right|_{\rm sequence} = \Omega$$

This is equivalent to requiring the constancy of the horizon area of each black hole, by virtue of the First law of thermodynamics for binary black holes :

$$dM_{\rm ADM} = \Omega\, dJ + \frac{1}{8\pi}\left(\kappa_1\, dA_1 + \kappa_2\, dA_2\right)$$

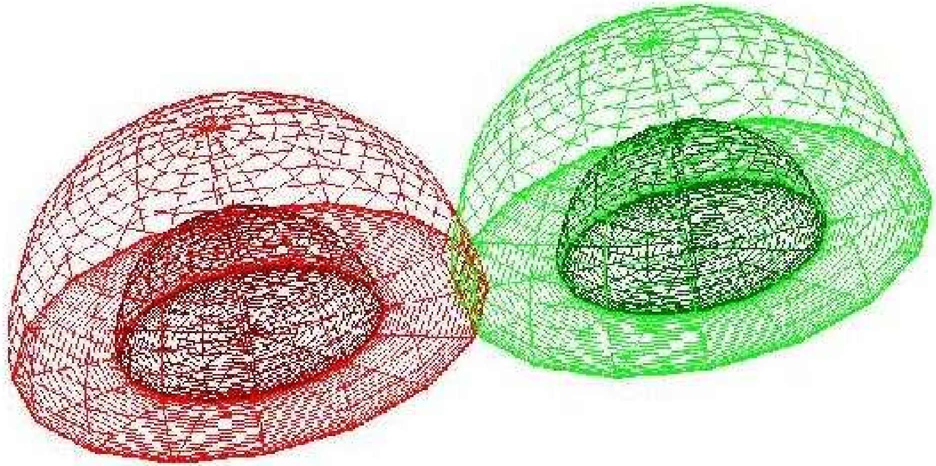
recently established by Friedman, Uryu & Shibata [PRD **65**, 064035 (2002)].

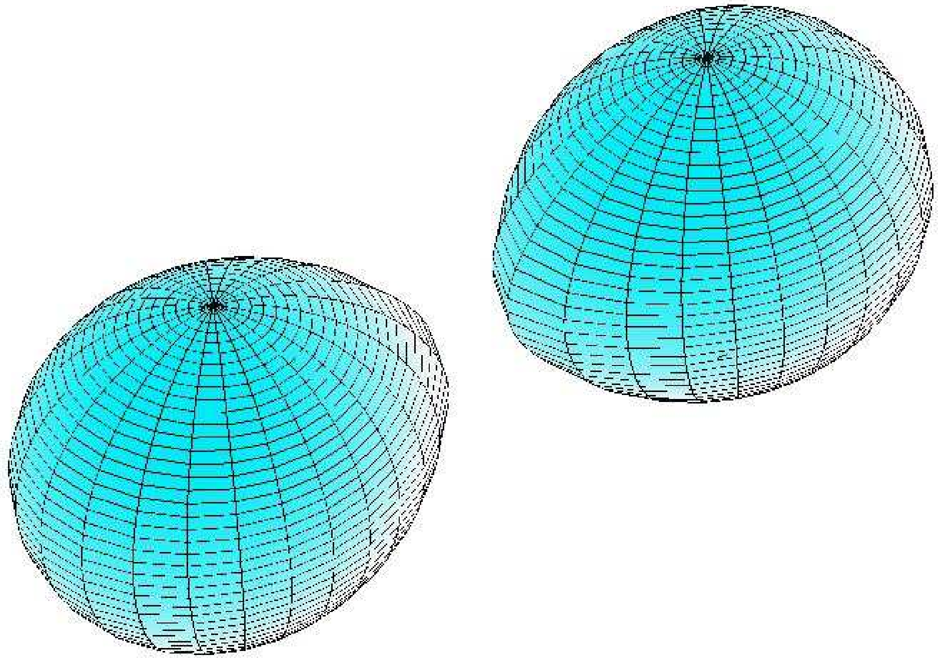
Note: Within the helical symmetry framework, a minimum in $M_{\rm ADM}$ along a sequence at fixed horizon area locates a change of orbital stability **(ISCO)** [Friedman, Uryu & Shibata, PRD **65**, 064035 (2002)].

An overview of the numerical techniques employed in Meudon

- Multidomain three-dimensional spectral method
- Spherical-type coordinates (r, θ, φ)
- Expansion functions: r : Chebyshev; θ : cosine/sine or associated Legendre functions; φ : Fourier
- Domains $=$ spherical shells $+$ 1 nucleus (contains $r = 0$)
- Entire space $(\mathbb{R}^3)$ covered: compactification of the outermost shell
- Adaptative coordinates : domain decomposition with spherical topology
- Multidomain PDEs: patching method (strong formulation)
- Numerical implementation: C++ codes based on LORENE

Domain decomposition





[Taniguchi, Gourgoulhon & Bonazzola, Phys. Rev. D **64**, 064012 (2001)]

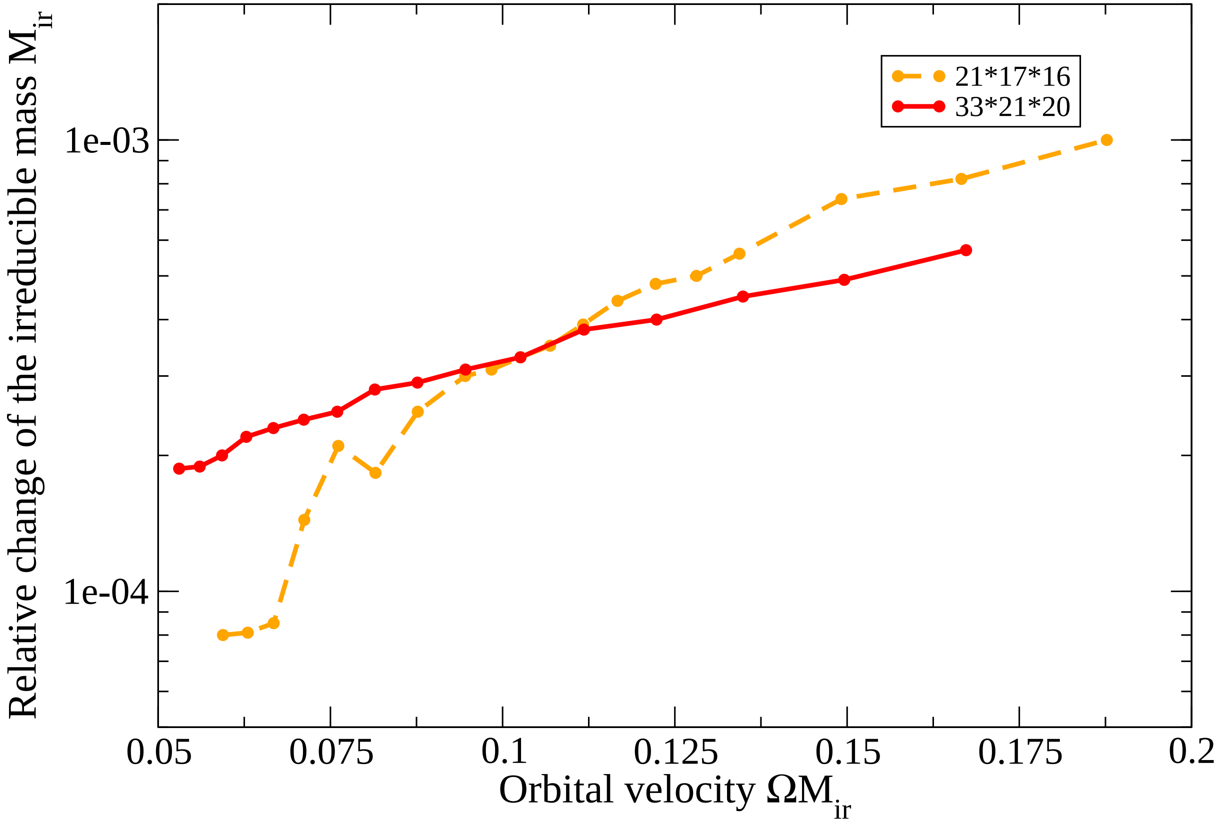
Double domain decomposition

Surface fitted coordinates:

$F_0(\theta, \varphi)$ and $G_0(\theta, \varphi)$ chosen so that

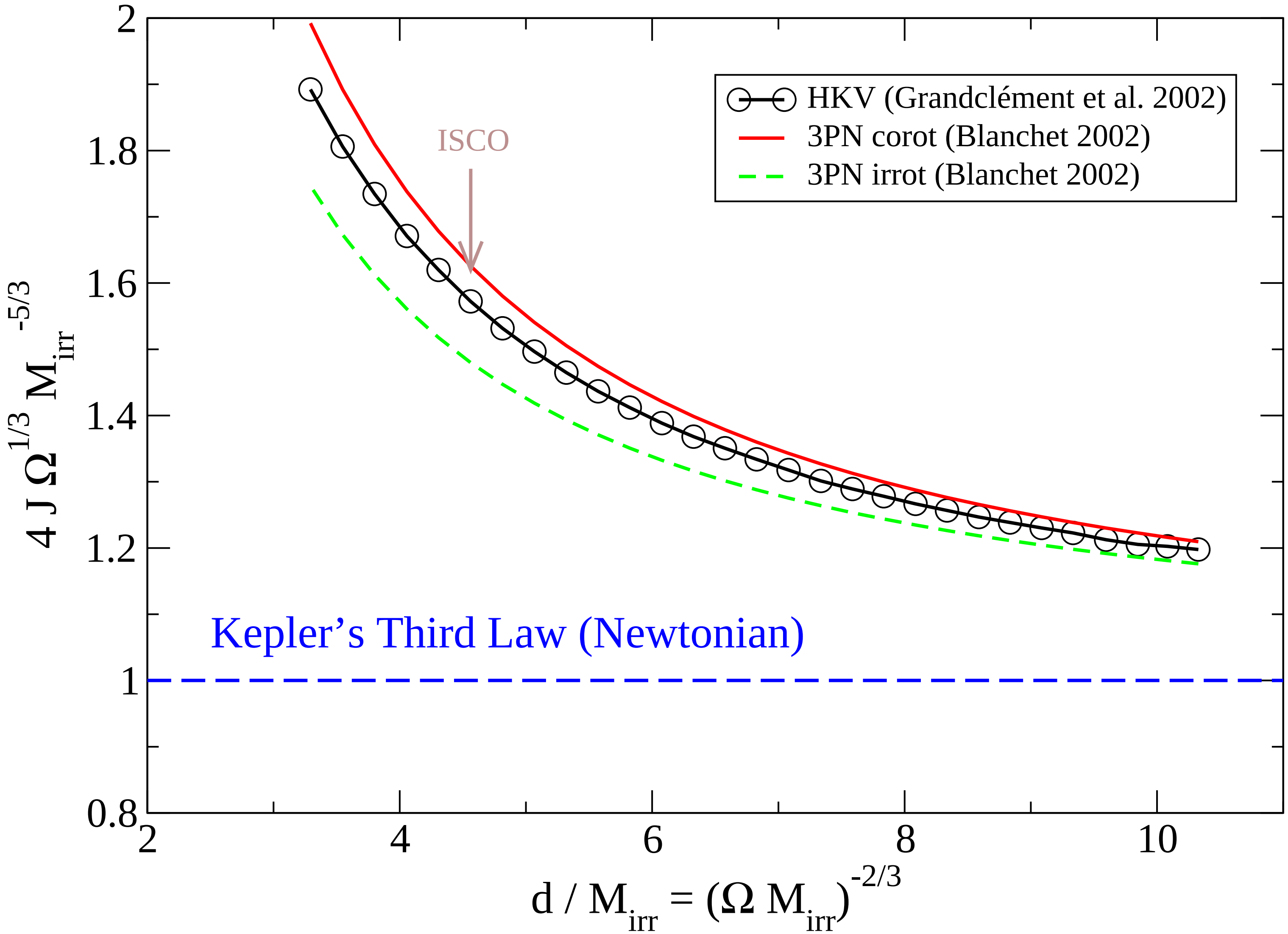
$\xi = 1 \Leftrightarrow$ surface of the star

Test for binary BH : conservation of the horizon area along a sequence



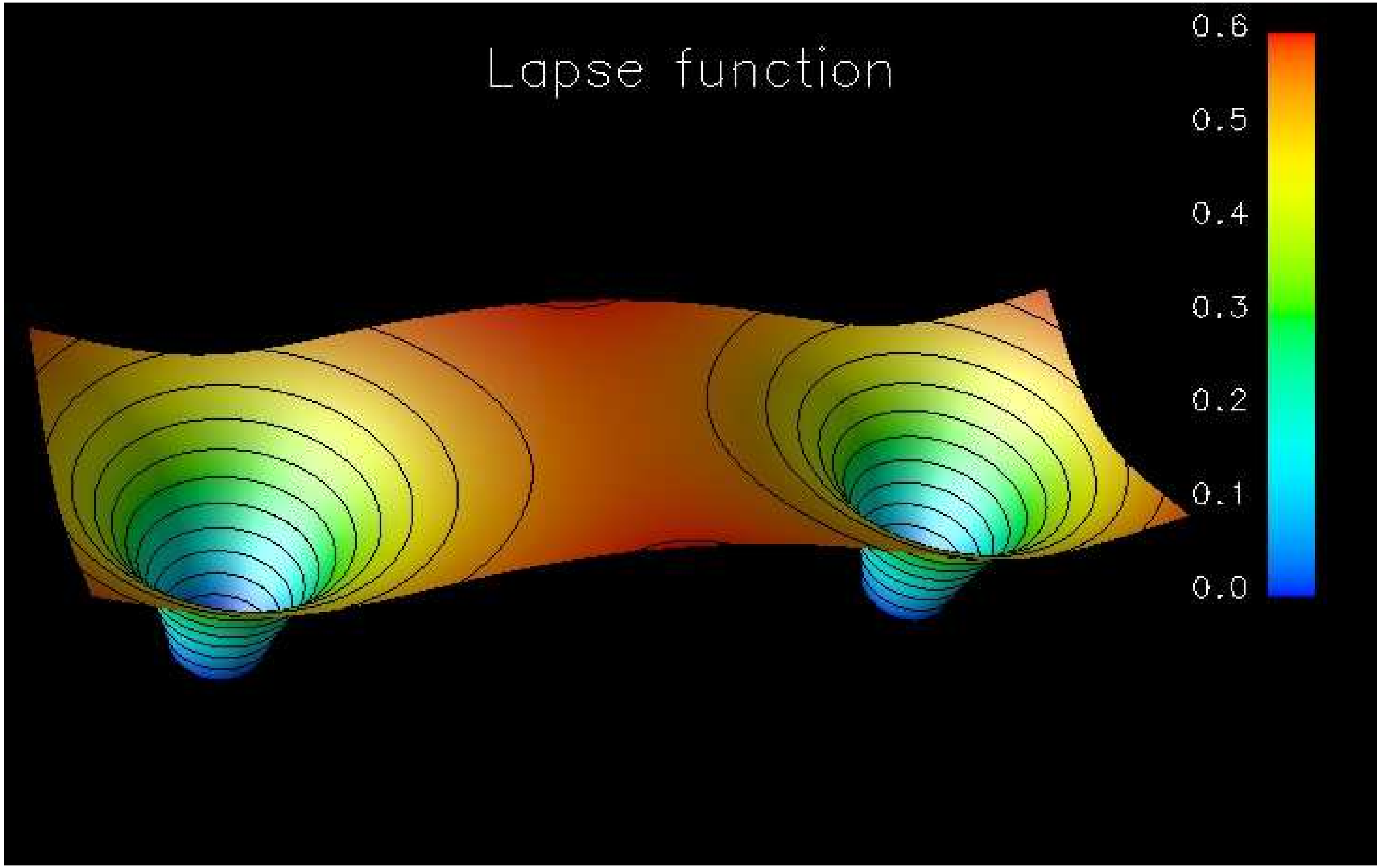
Relative change of the horizon area along an evolutionary sequence

Test for binary BH: recovering Kepler's third law



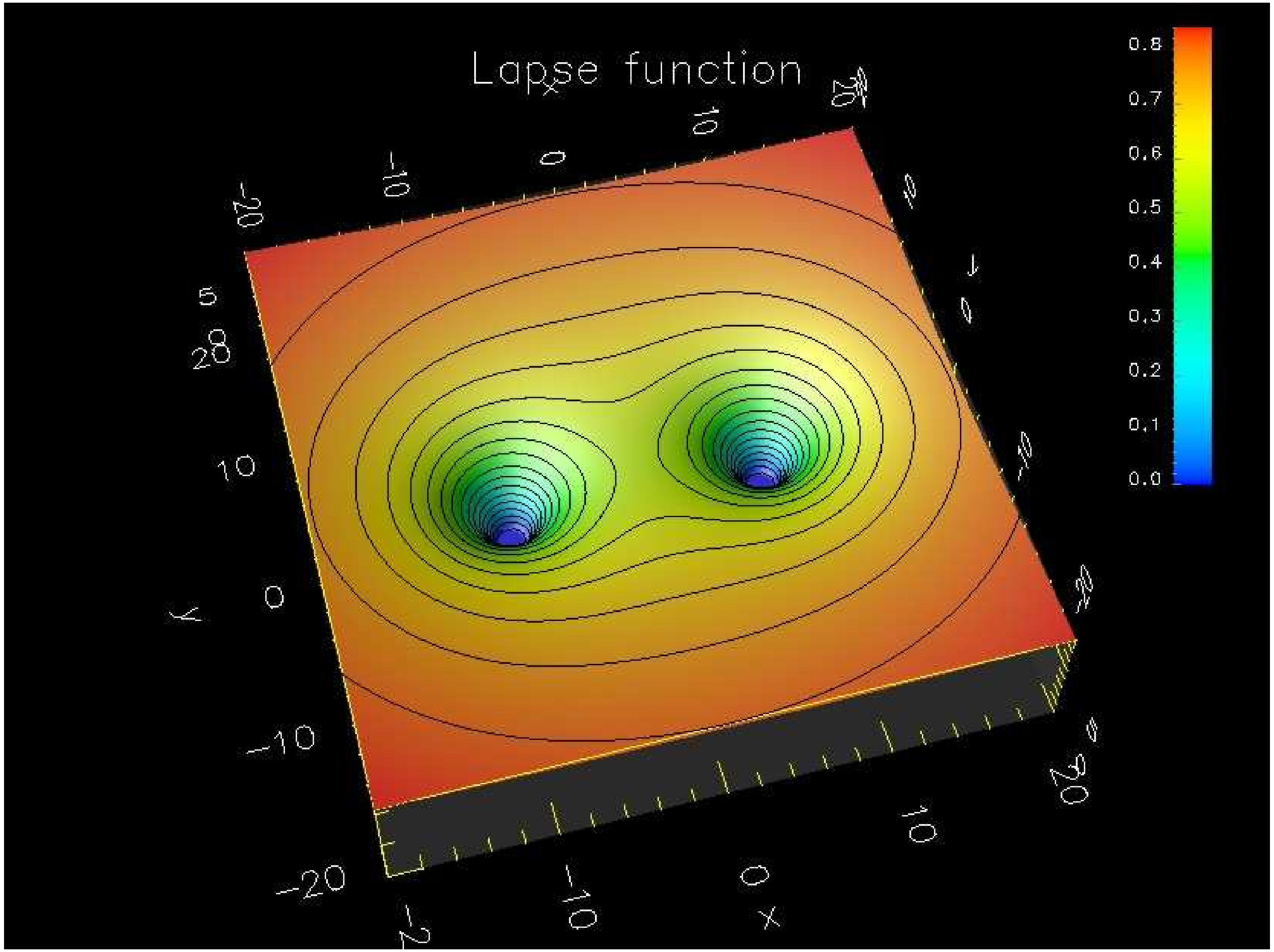
Check of the determination of Ω at large separation.

ISCO configuration



[Grandclément, Gourgoulhon, Bonazzola, PRD **65**, 044021 (2002)]

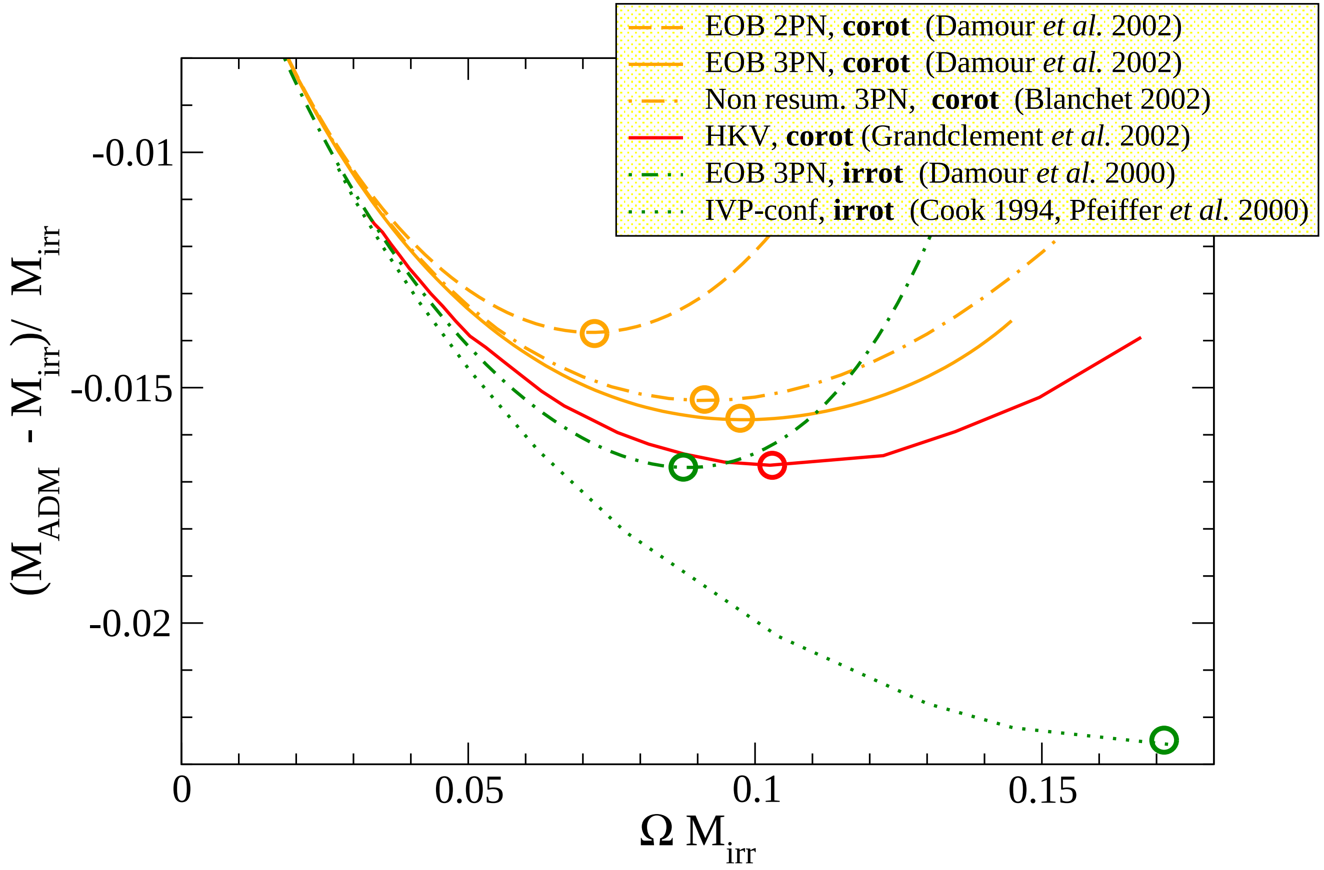
ISCO configuration



[Grandclément, Gourgoulhon, Bonazzola, PRD **65**, 044021 (2002)]

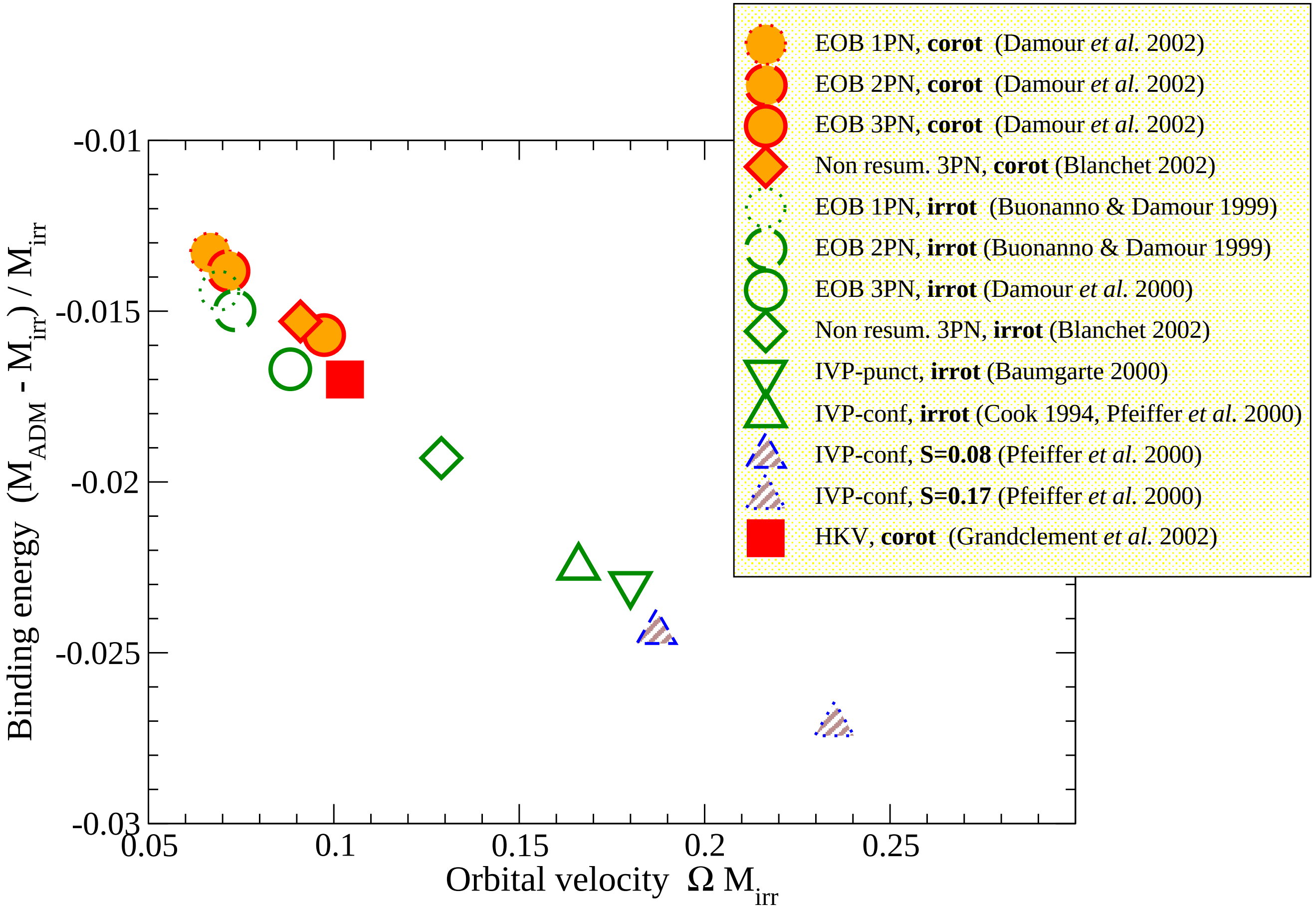
Comparison with Post-Newtonian computations

Binding energy along an evolutionary sequence of equal-mass binary black holes



[Damour, Gourgoulhon, Grandclément, PRD **66**, 024007 (2002)]

Location of the ISCO

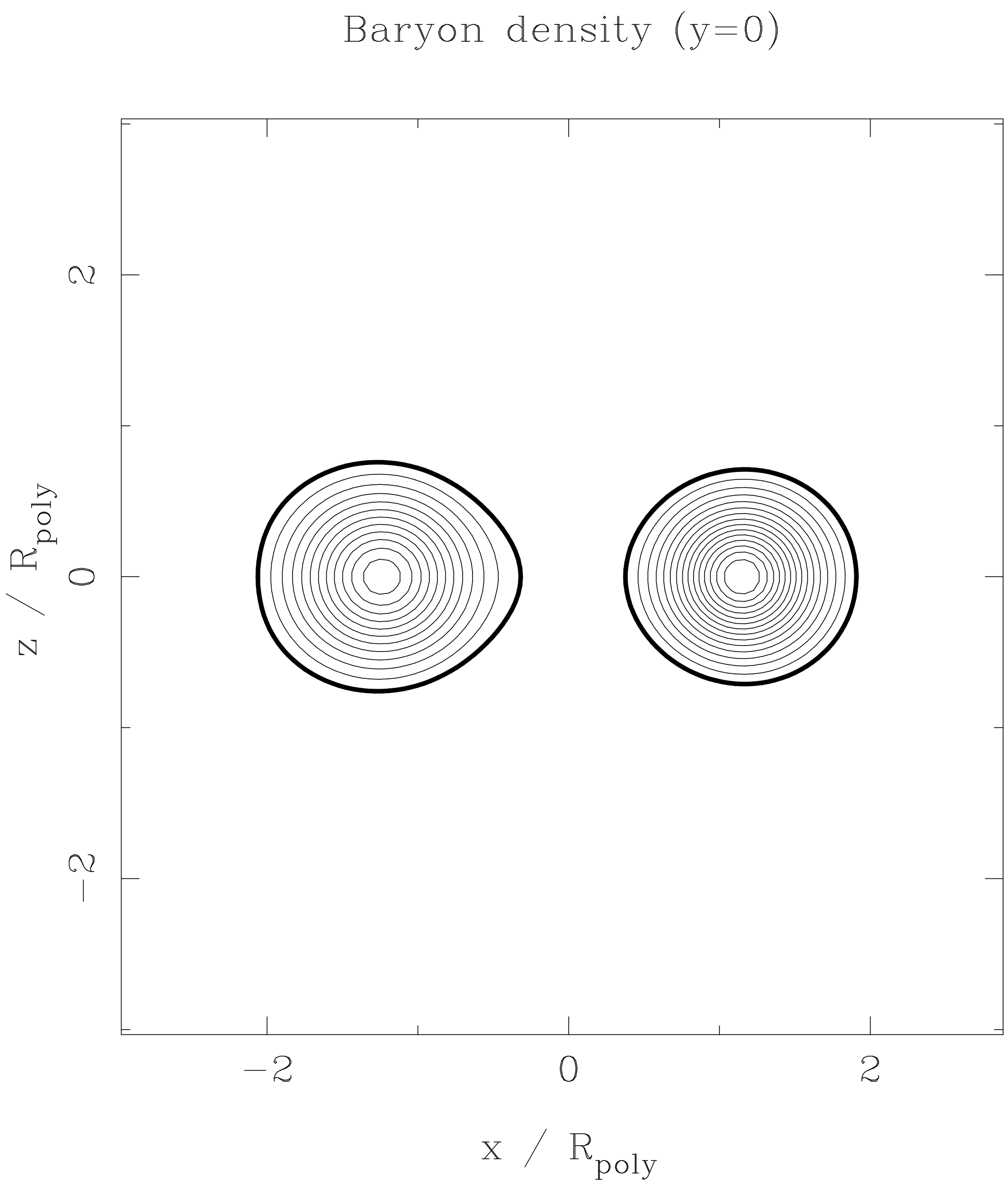


Gravitational wave frequency:

$$f = 320 \, \frac{\Omega M_{\rm ir}}{0.1} \, \frac{20 \, M_\odot}{M_{\rm ir}} \, \text{Hz}$$

[Damour, Gourgoulhon, Grandclément, PRD **66**, 024007 (2002)]

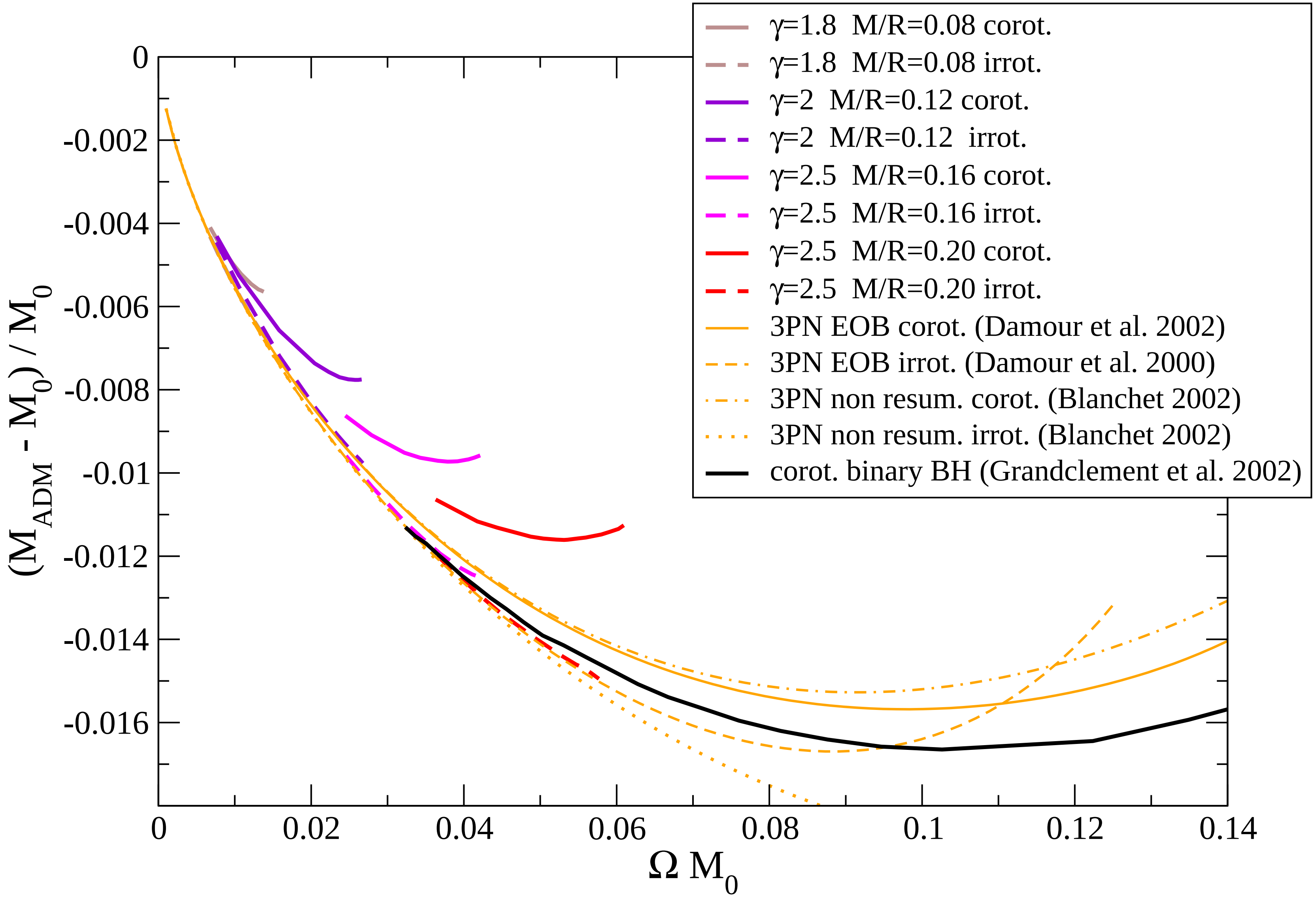
Results for binary NS



Isocontour of baryon density for an irrotational binary system constructed upon a polytropic EOS with $\gamma = 2$. The compactness of the left star is $M/R = 0.14$ and that of the right star is $M/R = 0.16$

[Taniguchi & Gourgoulhon, PRD **66**, 104019 (2002)]

Comparing binary NS and binary BH sequences



[Taniguchi & Gourgoulhon, submitted (2003)]

Source of the discrepancy between CTT+BY+EP and CTS+HKV

CTT+BY+EP = Conformal Transverse Traceless decomposition of the constraints + Bowen-York extrinsic curvature + Effective Potential determination of the orbits

CTS+HKV = Conformal Thin Sandwich decomposition of the constraints + Helical Killing Vector

Recall : both CTT+BY+EP and CTS+HKV methods employ a conformally flat 3-metric, so this cannot be the reason why CTT+BY+EP is far from post-Newtonian results.

Two main differences between CTT+BY+EP and CTS+HKV approaches:

- Criterion for a circular orbit and determination of the orbital angular velocity Ω
- Extrinsic curvature of the $t = \mathrm{const}$ hypersurface

The source of discrepancy lies in the extrinsic curvature

CTT+BY+EP definition of circular orbit and Ω lacks of rigor, due to the ad hoc definition of the binding energy. This is unavoidable, due to the intrinsic **3-dimensional** character of CTT+BY+EP :

no time in CTT+BY+EP $\Rightarrow$ no well-defined velocity !

On the contrary CTS+HKV is intrinsically **4-dimensional**, and its definition of Ω is unambiguous.

However, despite these differences, it turns out that the two ways of determining Ω for circular orbits yield the same result

- for irrotational black holes with the Bowen-York extrinsic curvature (Shibata 2002).

- for a simple analytical model of a spherical shell of collisionless particles (Skoge & Baumgarte 2002 [PRD **66**, 107501 (2002)])

$\Rightarrow$ **Main source of discrepancy: the extrinsic curvature**

Conclusions and future prospects

- Among the two methods CTT and CTS to solve the constraint equations, CTS is more appropriate to get quasiequilibrium initial data

- The classical Bowen-York extrinsic curvature does not represent well binary black holes in quasiequilibrium orbital motion

- The helical Killing vector approach results in very good agreement with post-Newtonian computations

- Next computational step: relaxing the conformal flatness hypothesis, while keeping the helical symmetry

- Also for future work: implement new inner boundary conditions (instead of the isometry condition), such as apparent horizon boundary [Maxwell, gr-qc/0307117], [Dain, gr-qc/0308009] $\Longrightarrow$ connection with dynamical horizons