DIMENSIONAL REGULARIZATION OF THE GRAVITATIONAL INTERACTION OF POINT MASSES

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Why Dirac deltas?

• **Effacement principle** (Damour, 1983): dimensions and internal structure of the **compact** nonrotating bodies enter their equations of motion (EOM) only at the 5PN order.

• EOM **up to 2.5PN order** were derived both for point particles and extended body models—the results are compatible.

• Point particles together with dimensional regularization give **unique** EOM up to 3PN order; these equations respect global Poincaré invariance.

• **Buchdahl's limit**: no contiunous transition exists from extended bodies to black holes; it seems reasonable to use δs in studying PN dynamics of binary black holes.

1 ADM formalism for N point-mass systems in d space dimensions

1.1 Reduced Hamiltonian

Asymptotically flat spacetime,
asymptotically Minkowskian reference frame.

Particle labels: $a, b, \ldots = 1, \ldots, N$.

$$\mathbf{r}_a := \mathbf{x} - \mathbf{x}_a, \quad r_a := |\mathbf{r}_a|, \quad \mathbf{n}_a := \mathbf{r}_a / r_a$$

For $a \neq b$: $\mathbf{r}_{ab} := \mathbf{x}_a - \mathbf{x}_b, \quad r_{ab} := |\mathbf{r}_{ab}|, \quad \mathbf{n}_{ab} := \mathbf{r}_{ab} / r_{ab}$.

Matter variables

$$\begin{aligned} \mathbf{x}_a &= (x_a^1, x_a^2, x_a^3), \\ \mathbf{p}_a &= (p_{a1}, p_{a2}, p_{a3}), \end{aligned} \qquad a = 1, \ldots, N. \tag{1}$$

Field variables

$$\begin{aligned} g_{ij} &:= {}^4 g_{ij}, \\ \pi^{ij} &:= \sqrt{g}(K^{ij} - g^{ij} g^{kl} K_{kl}), \end{aligned} \tag{2}$$

K_{ij} is the extrinsic curvature of the hypersurface $x^0 = const.$

The ADM constraint equations for N-point-mass system (in units $c = 16\pi G_{d+1} = 1$):

$$\boxed{\begin{aligned} \sqrt{g}\,R - \frac{1}{\sqrt{g}}\left(g_{ik}\,g_{j\ell}\,\pi^{ij}\,\pi^{k\ell} - \frac{1}{d-1}\,(g_{ij}\,\pi^{ij})^2\right) &= h, \quad (3a) \\ -2\left(\pi^{ij}{}_{,j} + \Gamma^i_{jk}\pi^{jk}\right) &= h^i, \quad (3b) \end{aligned}}$$

$$h = \sum_{a=1}^{N} \sqrt{\boxed{g_a^{ij}}\, p_{ai} p_{aj} + m_a^2}\; \delta(\mathbf{x} - \mathbf{x}_a)\,, \tag{4a}$$

$$h^i = \sum_{a=1}^{N} \boxed{g_a^{ij}}\, p_{aj}\, \delta(\mathbf{x} - \mathbf{x}_a)\,, \tag{4b}$$

$g_a^{ij} \equiv g^{ij}(\mathbf{x}_a)$ is perturbatively unambigously defined and finite (at least up to the 3PN order).

Eqs. (4) are derived from the energy momentum tensor

$$T^{\alpha\beta} = \sum_{a=1}^{N} m_a \int_{-\infty}^{+\infty} \frac{u^\alpha u^\beta}{\sqrt{-\det(^4 g_{\mu\nu})}} \delta(x^\mu - z^\mu(\tau_a))\, d\tau_a. \tag{5}$$

The ADM transverse-traceless (TT) gauge:

$$g_{ij} = \left(1 + \frac{d-2}{4(d-1)}\phi\right)^{4/(d-2)} \delta_{ij} + h_{ij}^{\mathrm{TT}}, \tag{6a}$$

$$\pi^{ii} = 0. \tag{6b}$$

$$\pi^{ij} = \tilde{\pi}^{ij} + \pi^{ij\mathrm{TT}}, \tag{7a}$$

$$\tilde{\pi}^{ij} = \partial_i V^j + \partial_j V^i - \frac{2}{d}\delta^{ij}\,\partial_k V^k. \tag{7b}$$

If the constraint equations and the gauge conditions are satisfied, the Hamiltonian can be put into its *reduced* form:

$$\boxed{H\left[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\mathrm{TT}}, \pi^{ij\mathrm{TT}}\right] = -\int \mathrm{d}^d x\, \Delta\phi\left[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\mathrm{TT}}, \pi^{ij\mathrm{TT}}\right].} \tag{8}$$

The equations of motion for the particles:

$$\dot{\mathbf{p}}_a = -\frac{\partial H}{\partial \mathbf{x}_a}, \quad \dot{\mathbf{x}}_a = \frac{\partial H}{\partial \mathbf{p}_a}, \quad a = 1, \ldots, N. \tag{9}$$

Evolution equations for the field degrees of freedom:

$$\frac{\partial}{\partial t}\pi^{ij\mathrm{TT}} = -\frac{\delta H}{\delta h_{kl}^{\mathrm{TT}}}, \qquad \frac{\partial}{\partial t}h_{ij}^{\mathrm{TT}} = \frac{\delta H}{\delta \pi^{kl\mathrm{TT}}}. \tag{10}$$

1.2 Post-Newtonian expansion of the reduced two-point-mass Hamiltonian

$\underline{N = 2}$

Zeroth order approximation: Newtonian gravity.
nPN order: corrections of order

$$\left(\frac{v}{c}\right)^{2n} \sim \left(\frac{Gm}{r\,c^2}\right)^{n}$$

to the Newtonian gravity.

Expansion of the functions in $1/c$ (the numbers within parentheses denote the order in $1/c$):

$$\phi = \phi_{(2)} + \phi_{(4)} + \dots\,, \tag{11a}$$

$$V^i = V^i_{(3)} + V^i_{(5)} + \dots\,, \tag{11b}$$

$$h^{\mathrm{TT}}_{ij} = h^{\mathrm{TT}}_{(4)ij} + h^{\mathrm{TT}}_{(5)ij} + \dots\,, \tag{11c}$$

$$\pi^{ij\mathrm{TT}} = \pi^{ij\mathrm{TT}}_{(5)} + \pi^{ij\mathrm{TT}}_{(6)} + \dots\,, \tag{11d}$$

leads to the PN expansion of the Hamiltonian:

$$\begin{aligned} H =& (m_1 + m_2)\,c^2 + H_{\mathrm{N}} + \frac{1}{c^2}\,H_{1\mathrm{PN}} + \frac{1}{c^4}\,H_{2\mathrm{PN}} \\ &+ \frac{1}{c^5}\,H_{2.5\mathrm{PN}} + \frac{1}{c^6}H_{3\mathrm{PN}} + \frac{1}{c^7}H_{3.5\mathrm{PN}} \\ &+ \mathcal{O}\left(\frac{1}{c^8}\right). \end{aligned} \tag{12}$$

3PN-accurate *conservative* two-point-mass *matter* Hamiltonian:

$$\begin{aligned} H(\mathbf{x}_a, \mathbf{p}_a) =& (m_1 + m_2)c^2 + H_{\mathsf{N}}(\mathbf{x}_a, \mathbf{p}_a) \\ &+ \frac{1}{c^2} H_{1\mathsf{PN}}(\mathbf{x}_a, \mathbf{p}_a) + \frac{1}{c^4} H_{2\mathsf{PN}}(\mathbf{x}_a, \mathbf{p}_a) \\ &+ \frac{1}{c^6} H_{3\mathsf{PN}}(\mathbf{x}_a, \mathbf{p}_a)\,. \end{aligned} \tag{13}$$

Equations of motion of the particles:

$$\dot{\mathbf{x}}_a = \frac{\partial H}{\partial \mathbf{p}_a}, \quad \dot{\mathbf{p}}_a = -\frac{\partial H}{\partial \mathbf{x}_a}. \tag{14}$$

The 3PN-accurate Hamiltonian can be expressed in terms of the six functions: $\phi_{(2)}$, $\phi_{(4)}$, $V^i_{(3)}$, $h^{\mathrm{TT}}_{(4)ij}$, $S_{(4)}$, and $S_{(4)ij}$.

They satisfy the equations $\Big(\delta_a \equiv \delta(\mathbf{x}-\mathbf{x}_a)\Big)$:

$$\Delta\,\phi_{(2)} = -\sum_a m_a\,\delta_a\,, \tag{15a}$$

$$\Delta\,\phi_{(4)} = -\frac{1}{2}\sum_a \frac{\mathbf{p}_a^2}{m_a}\delta_a + \frac{d-2}{4(d-1)}\phi_{(2)}\sum_a m_a\delta_a\,, \tag{15b}$$

$$\Delta\,V^i_{(3)} + \left(1-\frac{2}{d}\right)\,\partial_{ij}V^j_{(3)} = -\frac{1}{2}\sum_a p_{ai}\,\delta_a\,, \tag{15c}$$

$$\Delta h^{\mathrm{TT}}_{(4)ij} = \left(-\sum_a \frac{p_{ai}p_{aj}}{m_a}\delta_a - \frac{d-2}{2(d-1)}\phi_{(2),i}\phi_{(2),j}\right)^{\mathrm{TT}}, \tag{15d}$$

$$\Delta S_{(4)} = \sum_a \frac{\mathbf{p}_a^2}{m_a}\delta_a\,, \tag{15e}$$

$$\Delta S_{(4)ij} = \sum_a \frac{p_{ai}p_{aj}}{m_a}\delta_a\,. \tag{15f}$$

One uses the relations:

$$\Delta^{-1}\delta_a = -k\, r_a^{2-d}, \quad k := \Gamma\left(\frac{d-2}{2}\right)/(4\pi^{\frac{d}{2}})\,, \tag{16a}$$

$$\Delta^{-1} r_a^{\lambda} = \frac{r_a^{\lambda+2}}{(\lambda+2)(\lambda+d)}\,. \tag{16b}$$

E.g., $\quad \phi_{(2)} = -\sum_a m_a\, \Delta^{-1}\delta_a = k \sum_a m_a r_a^{2-d}\,.$

$\phi_{(2)}$, $\phi_{(4)}$, $V^i_{(3)}$, $S_{(4)}$, $S_{(4)ij}$, and the quadratic in momenta part of $h^{\mathrm{TT}}_{(4)ij}$ can be found.

2 Regularization of the 3PN two-point-mass Hamiltonian in $d = 3$ space dimensions

2.1 Regularization prescriptions and ambiguity of the regularization results

$$H(\mathbf{x}_a, \mathbf{p}_a) = \int \mathrm{d}^3x \, \mathcal{H}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a), \tag{17a}$$

$$\mathcal{H}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) = \mathcal{H}_c^{(D)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) + \mathcal{H}_f^{(D)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) + \partial_i D^i(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a), \tag{17b}$$

$\partial_i D^i$ gives no contribution to the H.

Structure of *contact* $\mathcal{H}_c^{(D)}$ and *field-like* $\mathcal{H}_f^{(D)}$ terms:

$$\mathcal{H}_c^{(D)} = \sum_a S_a(\mathbf{x}; \mathbf{x}_b, \mathbf{p}_b) \, \delta(\mathbf{x} - \mathbf{x}_a), \tag{18a}$$

$$\mathcal{H}_f^{(D)} = \sum_{\boldsymbol{\ell}} c_{\boldsymbol{\ell}} \, (\mathbf{n}_1 \cdot \mathbf{p}_1)^{\ell_1} \, (\mathbf{n}_2 \cdot \mathbf{p}_1)^{\ell_2} \, (\mathbf{n}_1 \cdot \mathbf{p}_2)^{\ell_3} \, (\mathbf{n}_2 \cdot \mathbf{p}_2)^{\ell_4} \times r_1^{\ell_5} \, r_2^{\ell_6} \, (r_1 + r_2 + r_{12})^{\ell_7}. \tag{18b}$$

Contact terms: Hadamard's "partie finie"

$$\int \mathrm{d}^3x \, S_a(\mathbf{x}; \mathbf{x}_b, \mathbf{p}_b) \, \delta(\mathbf{x} - \mathbf{x}_a) \equiv \lim_{\mathbf{x} \to \mathbf{x}_a} S_a(\mathbf{x}; \mathbf{x}_b, \mathbf{p}_b)$$

$$:= \mathrm{Pf}_a \, S_a(\mathbf{x}; \mathbf{x}_b, \mathbf{p}_b); \quad \text{(19a)}$$

$$S_a(\mathbf{x}_a + r_a \mathbf{n}_a; \mathbf{x}_b, \mathbf{p}_b) = \sum_{m=-M}^{\infty} a_m(\mathbf{n}_a) \, r_a^m \,, \qquad \text{(19b)}$$

$$\mathrm{Pf}_a \, S_a(\mathbf{x}; \mathbf{x}_b, \mathbf{p}_b) := \frac{1}{4\pi} \oint d\Omega_a \, a_0(\mathbf{n}_a). \qquad \text{(19c)}$$

Important property: in general,

$$\mathrm{Pf}_a(f_1 f_2 \cdots) \neq (\mathrm{Pf}_a \, f_1)(\mathrm{Pf}_a \, f_2) \cdots \,. \qquad \text{(19d)}$$

Field-like terms: Riesz's analytic continuation

Let us consider singular integral

$$I := \int_{\mathbb{R}^3} \mathrm{d}^3x \, F(\mathbf{x}; \mathbf{x}_1, \mathbf{x}_2). \tag{20}$$

$$\mathbb{R}^3 = V_1 \cup V_2 \cup V \quad (\mathbf{x}_1 \in V_1,\ \mathbf{x}_2 \in V_2,\ V_1 \cap V_2 = \emptyset)$$

$$I_1(\varepsilon_1) := \int_{V_1} \mathrm{d}^3x \left(\frac{r_1}{l_1}\right)^{\varepsilon_1} F(\mathbf{x}; \mathbf{x}_1, \mathbf{x}_2), \tag{21}$$

where l_1 is a regularization length scale.

Some integrals gives rise to a pole, as $\varepsilon_1 \to 0$:

$$I_1(\varepsilon_1) = Z_1 \left(\frac{1}{\varepsilon_1} + \ln\left(\frac{R_1}{l_1}\right)\right) + A_1 + \mathcal{O}(\varepsilon_1), \tag{22}$$

where R_1 is a length scale associated with the choice of V_1.

In computation of the 3PN ADM Hamiltonian all pole terms exactly cancel:

$$\boxed{\sum Z_1 = 0.} \tag{23}$$

Generalized Riesz formula

$$\int \mathrm{d}^3x \, r_1^\alpha \, r_2^\beta (r_1 + r_2 + r_{12})^\gamma = 2\pi \, r_{12}^{\alpha+\beta+\gamma+3}$$

$$\times \Big(B(\alpha+2, \beta+2) B_{1/2}(-\alpha-\beta-\gamma-4, \alpha+\beta+4)$$

$$- B(-\alpha-\beta-4, \beta+2) B_{1/2}(-\alpha-\gamma-2, \alpha+2)$$

$$- B(-\alpha-\beta-4, \alpha+2) B_{1/2}(-\beta-\gamma-2, \beta+2) \Big) \,,$$

where B is the beta function,
$B_{1/2}$ is the incomplete beta function:

$$B_{1/2}\,(\alpha, \beta) = \frac{1}{\alpha 2^\alpha} \, {}_2F_1\Big(1-\beta, \alpha; \alpha+1; \frac{1}{2}\Big) \,,$$

${}_2F_1$ is the Gauss hypergeometric function.

$$H = \int \mathrm{d}^3x \, \mathcal{H}, \tag{24a}$$

$$\mathcal{H} = \mathcal{H}_c^{(D)} + \mathcal{H}_f^{(D)} + \partial_i D^i. \tag{24b}$$

When operating by parts
(which changes D^i, $\mathcal{H}_c^{(D)}$, and $\mathcal{H}_f^{(D)}$),
the regularized value

$$H_{\mathrm{reg}}^{(D)} = H_{c\,\mathrm{reg}}^{(D)} + H_{f\,\mathrm{reg}}^{(D)} \tag{25}$$

is found to change.

$$H(\mathbf{x}_a, \mathbf{p}_a) = (m_1 + m_2)\, c^2 + H_{\mathrm{N}}(\mathbf{x}_a, \mathbf{p}_a) + \frac{1}{c^2}\, H_{1\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) + \frac{1}{c^4}\, H_{2\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) + \frac{1}{c^6}\, H_{3\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a), \tag{26}$$

$$H_{\mathrm{N}}(\mathbf{x}_a, \mathbf{p}_a) = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} - \frac{G m_1 m_2}{r_{12}}, \quad \text{etc.} \tag{27}$$

$$\boxed{H_{3\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) = H_{3\mathrm{PN}}^{\mathrm{reg}}(\mathbf{x}_a, \mathbf{p}_a) + H_{3\mathrm{PN}}^{\mathrm{kinetic}}(\mathbf{x}_a, \mathbf{p}_a) + H_{3\mathrm{PN}}^{\mathrm{static}}(\mathbf{x}_a, \mathbf{p}_a),} \tag{28}$$

$$H_{3\mathrm{PN}}^{\mathrm{kinetic}}(\mathbf{x}_a, \mathbf{p}_a) = \boxed{\omega_{\mathrm{kinetic}}}\, \frac{G^3\, m_1\, m_2}{2\, r_{12}^3} \times \left[\mathbf{p}_1^2 - 3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2 + \left(1 \leftrightarrow 2\right)\right], \tag{29a}$$

$$H_{3\mathrm{PN}}^{\mathrm{static}}(\mathbf{x}_a, \mathbf{p}_a) = \boxed{\omega_{\mathrm{static}}}\, \frac{G^4\, m_1^2\, m_2^2\, (m_1 + m_2)}{r_{12}^4}. \tag{29b}$$

2.2 Poincaré invariance of the ADM dynamics and the determination of the kinetic ambiguity parameter

Poisson brackets structure on the two-point-mass phase space:

$$\{A(\mathbf{x}_a, \mathbf{p}_a), B(\mathbf{x}_a, \mathbf{p}_a)\} := \sum_{a=1}^{2} \sum_{i=1}^{3} \left(\frac{\partial A}{\partial x_a^i} \frac{\partial B}{\partial p_{ai}} - \frac{\partial A}{\partial p_{ai}} \frac{\partial B}{\partial x_a^i} \right). \tag{30}$$

One requires the existence of generators of the Poincaré algebra realized as *functions* $P^\mu(\mathbf{x}_a, \mathbf{p}_a)$ and $J^{\mu\nu}(\mathbf{x}_a, \mathbf{p}_a)$:

$$\{P^\mu, P^\nu\} = 0, \tag{31a}$$

$$\{P^\mu, J^{\rho\sigma}\} = -\eta^{\mu\rho} P^\sigma + \eta^{\mu\sigma} P^\rho, \tag{31b}$$

$$\{J^{\mu\nu}, J^{\rho\sigma}\} = \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\mu\sigma} J^{\nu\rho} - \eta^{\nu\rho} J^{\mu\sigma} + \eta^{\nu\sigma} J^{\mu\rho}, \tag{31c}$$

where $\eta^{\mu\nu} := \mathrm{diag}\,(-1, +1, +1, +1)$.

The generators P^μ and $J^{\mu\nu}$ are decomposed as:

$$P^0 \equiv H(\mathbf{x}_a, \mathbf{p}_a) \text{ (including the rest-mass contribution),} \tag{32a}$$

$$P^i \equiv \text{three-momentum,} \tag{32b}$$

$$J^i \equiv \frac{1}{2}\varepsilon^{ik\ell} J_{k\ell} \quad \text{(angular momentum),} \tag{32c}$$

$$K^i \equiv J^{i0} \quad \text{(boost vector).} \tag{32d}$$

One further decomposes the boost vector $\mathbf{K}$:

$$K^i(\mathbf{x}_a, \mathbf{p}_a; t) \equiv G^i(\mathbf{x}_a, \mathbf{p}_a) - t\, P^i(\mathbf{x}_a, \mathbf{p}_a), \tag{33}$$

so that

$$\frac{dK^i}{dt} = \frac{\partial K^i}{\partial t} + \{K^i,\, H\} = -P^i + \{G^i,\, H\} = 0. \tag{34}$$

$$\{P_i, H\} = \{J_i, H\} = 0\,, \tag{35a}$$

$$\{J_i, P_j\} = \varepsilon_{ijk}\,P_k\,,\ \ \{J_i, J_j\} = \varepsilon_{ijk}\,J_k\,, \tag{35b}$$

$$\{J_i, G_j\} = \varepsilon_{ijk}\,G_k\,, \tag{35c}$$

$$\underline{\{G_i, H\} = P_i}\,, \tag{35d}$$

$$\underline{\{G_i, P_j\} = \frac{1}{c^2}\,H\,\delta_{ij}\,,} \tag{35e}$$

$$\underline{\{G_i, G_j\} = -\frac{1}{c^2}\,\varepsilon_{ijk}\,J_k\,.} \tag{35f}$$

The generators P_i and J_i are realized as

$$P_i(\mathbf{x}_a, \mathbf{p}_a) = \sum_a p_{ai}\,, \qquad J_i(\mathbf{x}_a, \mathbf{p}_a) = \sum_a \varepsilon_{ik\ell}\,x_a^k\,p_{a\ell}\,. \tag{36}$$

$H(\mathbf{x}_a, \mathbf{p}_a)$ is translationally and rotationally invariant:
Eqs. (35a) and (35b) are exactly satisfied.

Eq. (35c) will be exactly satisfied
if G_i is constructed as a three-vector from $\mathbf{x}_a$ and $\mathbf{p}_a$.

<u>There should exist a vector $G_i(\mathbf{x}_a, \mathbf{p}_a)$ satisfying relations (35d), (35e), and (35f).</u>

At the Newtonian level, equations

$$\{G_i, H\} = P_i\,, \tag{37a}$$

$$\{G_i, P_j\} = (m_1 + m_2)\,\delta_{ij}\,, \tag{37b}$$

$$\{G_i, G_j\} = 0\,, \tag{37c}$$

are satisfied by the vector

$$G^i_{\mathsf{N}}(\mathbf{x}_a, \mathbf{p}_a) := \sum_a m_a\, x^i_a\,. \tag{38}$$

Does there exist a 3PN-accurate vector G^i such that Eqs. (35d)–(35f) are fulfilled (within the 3PN accuracy)?

The method of undetermined coefficients

$$G^i(\mathbf{x}_a, \mathbf{p}_a) = \sum_{a=1}^{2} \Big[M_a(\mathbf{x}_b, \mathbf{p}_b)\, x_a^i + N_a(\mathbf{x}_b, \mathbf{p}_b)\, p_{ai} \Big], \quad \text{(39a)}$$

$$M_a = m_a + \frac{1}{c^2} M_a^{1\mathrm{PN}} + \frac{1}{c^4} M_a^{2\mathrm{PN}} + \frac{1}{c^6} M_a^{3\mathrm{PN}}, \quad \text{(39b)}$$

$$N_a = \frac{1}{c^2} N_a^{1\mathrm{PN}} + \frac{1}{c^4} N_a^{2\mathrm{PN}} + \frac{1}{c^6} N_a^{3\mathrm{PN}}. \quad \text{(39c)}$$

$M_a^{n\mathrm{PN}}$ and $N_a^{n\mathrm{PN}}$ are sums of scalar monomials of the form

$$c_{n_0 n_1 n_2 n_3 n_4 n_5}\, r_{12}^{-n_0} \times (\mathbf{p}_1^2)^{n_1} (\mathbf{p}_2^2)^{n_2} (\mathbf{p}_1 \cdot \mathbf{p}_2)^{n_3} (\mathbf{n}_{12} \cdot \mathbf{p}_1)^{n_4} (\mathbf{n}_{12} \cdot \mathbf{p}_2)^{n_5},$$

with positive integers $n_0, \ldots, n_5$.

One takes into account:

• dimensional analysis
(which constrains the possible values of $n_0, \ldots, n_5$);

• Euclidean covariance (including parity symmetry);

• time reversal symmetry
(M_a is even and N_a odd under $\mathbf{p}_a \to -\mathbf{p}_a$).

The ansatz yields a system of equations
for unknown coefficients c_n.

At the 1PN and 2PN levels
the solutions of these equations is *unique*.

At the 3PN level:

- the ansatz for $M_a^{\text{3PN}}$and N_a^{3PN} involves 78 unknown coefficients c_n and yields 138 equations to be satisfied;

- the quantity ω_{kinetic} enters the system of equations (ω_{static} drops out of the problem).

There exists a *unique* value of ω_{kinetic} for which the system of equations is compatible:

$$\boxed{\omega_{\text{kinetic}} = \frac{41}{24}\,.} \tag{40}$$

If $\omega_{\text{kinetic}} \neq 41/24$, the 3PN Hamiltonian does not admit a global Poincaré invariance.

If $\omega_{\text{kinetic}} = 41/24$, there is a *unique* solution.

2.3 Equivalence between the ADM-Hamiltonian and the harmonic-coordinates approaches

The ADM-Hamiltonian approach

ADM variables: $\mathbf{x}_a$, $\mathbf{p}_a$.

Equations of motion:

$$\dot{\mathbf{x}}_a = \frac{\partial H}{\partial \mathbf{p}_a}\,, \qquad \dot{\mathbf{p}}_a = -\frac{\partial H}{\partial \mathbf{x}_a}\,. \tag{41}$$

The harmonic-coordinates approach

(Blanchet, Faye, de Andrade)

Harmonic variables: $\mathbf{y}_a$, $\mathbf{v}_a \equiv \dot{\mathbf{y}}_a$.

Equations of motion:

$$\ddot{\mathbf{y}}_a = \mathbf{A}_a(\mathbf{y}_b, \mathbf{v}_b;\, y_1', y_2', \lambda)\,, \tag{42}$$

y_1' and y_2' are some regularization length scales (can be gauged away);

λ is the dimensionless regularization parameter.

The *necessary and sufficient* condition
for the transformation

$$\mathbf{y}_a(t) = \mathbf{Y}_a\big(\mathbf{x}_b(t), \mathbf{p}_b(t)\big)\,, \tag{43a}$$

$$\mathbf{v}_a(t) = \mathbf{V}_a\big(\mathbf{x}_b(t), \mathbf{p}_b(t)\big)\,, \tag{43b}$$

to map the ADM dynamics onto the harmonic one is:

$$\{\mathbf{Y}_a, H\} = \mathbf{V}_a, \quad \{\mathbf{V}_a, H\} = \mathbf{A}_a, \tag{44}$$

$$\Downarrow$$

$$\underline{\{\{\mathbf{Y}_a, H\}, H\} = \mathbf{A}_a\Big(\mathbf{Y}_b, \{\mathbf{Y}_b, H\}\Big).} \tag{45}$$

Eqs. (45) are equations for the two unknown functions $\mathbf{Y}_1(\mathbf{x}_b, \mathbf{p}_b)$ and $\mathbf{Y}_2(\mathbf{x}_b, \mathbf{p}_b)$.

<u>The method of undetermined coefficients</u>

The ansatz for $\mathbf{Y}_1$ and $\mathbf{Y}_2$ gives a *linear* system of 512 equations for the $2 \times 52 = 104$ unknown coefficients.

This system is compatible *if and only if* the ambiguity parameters ω_{static} and λ are related by

$$\lambda = -\frac{3}{11}\,\omega_{\text{static}} - \frac{1987}{3080}\,. \tag{46}$$

Then the solution is *unique*.

3 Dimensional regularization of the 3PN two-point-mass Hamiltonian

3.1 Finiteness of the Hamiltonian as $d \to 3$

$$\boxed{H_{\mathrm{3PN}}(d=3) := \lim_{d\to 3} H_{\mathrm{3PN}}(d),} \tag{47}$$

if no poles proportional to $1/(d-3)$ arise when $d \to 3$.

There exists **ten** terms T_A, $A = 1, \ldots, 10$, giving rise to poles as $d \to 3$.

Near $r_1 = 0$ they have all the structure:

$$T_A = k^4 \, m_1 \, m_2 \, r_1^{6-3d} \\ \times \Big(c_{A1} \, D(\mathbf{p}_1, \mathbf{p}_1) + c_{A2} \, (\mathbf{n}_1 \cdot \mathbf{p}_1) \, D(\mathbf{n}_1, \mathbf{p}_1) \\ + \Big[c_{A3} \, (\mathbf{n}_1 \cdot \mathbf{p}_1)^2 + c_{A4} \, (\mathbf{p}_1)^2 \Big] D(\mathbf{n}_1, \mathbf{n}_1) \Big), \tag{48}$$

where $c_{A1}, \ldots, c_{A4}$ are some d-dependent coefficients,

$$D(\mathbf{p}, \mathbf{q}) := \left(\partial_{ij} r_2^{2-d} \right)_{\mathbf{x}=\mathbf{x}_1} p^i \, q^j,$$

$$k := \Gamma\left(\frac{d-2}{2} \right) / (4\pi^{\frac{d}{2}}).$$

The "pole" contribution of the term T_A to the Hamiltonian $H_{3\mathsf{PN}}(d)$:

$$H_A^{\mathsf{loc\ sing\ log}}(d) := \int_{B(\mathbf{x}_1,\ell_1)} \mathrm{d}^d x\, T_A$$

$$= -\frac{1}{2}\, \Omega_d\, k^4\, m_1\, m_2\, D(\mathbf{p}_1,\mathbf{p}_1)\, \ell_1^{-2(d-3)}\, \frac{c_A(d)}{d-3}, \qquad (49)$$

where Ω_d is the area of the unit sphere in d dimensions.

One expands the coefficients $c_A(d)$ in powers of $\varepsilon \equiv d-3$:

$$c_A(d) = c_A(3) + \varepsilon\, c'_A(3) + \mathcal{O}(\varepsilon^2)\,. \qquad (50)$$

A	c_A
1	$\frac{21}{512} + \frac{1013}{10240}\,\varepsilon$
2	$-\frac{7}{128} - \frac{1649}{7680}\,\varepsilon$
3	$\frac{7}{2560} + \frac{601}{153600}\,\varepsilon$
4	$\frac{1}{256} + \frac{53}{5120}\,\varepsilon$
5	$-\frac{1}{32} - \frac{33}{640}\,\varepsilon$
6	$\frac{49}{1280} + \frac{5467}{76800}\,\varepsilon$
7	$\frac{1}{16} + \frac{15}{64}\,\varepsilon$
8	$-\frac{1}{16} - \frac{7}{64}\,\varepsilon$
9	$-\frac{21}{2560} - \frac{5423}{153600}\,\varepsilon$
10	$\frac{21}{2560} + \frac{2903}{153600}\,\varepsilon$

The total pole part of $H_{3\mathrm{PN}}(d)$ vanishes as $d \to 3$:

$$\boxed{\sum_{A=1}^{10} c_A(3) = 0\,.} \tag{51}$$

3.2 Removing ambiguities of the 3-dimensional regularization results

Let $H_{\text{3PN 3D reg}}$ be the 3PN Hamiltonian obtained by Riesz-implemented Hadamard regularization performed in $d = 3$ space dimensions.

The correct 3PN Hamiltonian can be computed as

$$\lim_{d\to 3} H_{\text{3PN}}(d) = H_{\text{3PN 3D reg}} + \Delta H_{\text{3PN}}, \tag{52}$$

where

$$\Delta H_{\text{3PN}} = \lim_{d\to 3} H_{\text{3PN}}(d) - H_{\text{3PN 3D reg}} \tag{53a}$$

$$= \lim_{d\to 3} H^{\text{loc}}_{\text{3PN}}(d) - H^{\text{loc}}_{\text{3PN 3D reg}} \tag{53b}$$

$$= \lim_{d\to 3} H^{\text{loc sing}}_{\text{3PN}}(d) - H^{\text{loc sing}}_{\text{3PN 3D reg}} \tag{53c}$$

$$= \lim_{d\to 3} H^{\text{loc sing log}}_{\text{3PN}}(d) - H^{\text{loc sing log}}_{\text{3PN 3D reg}}. \tag{53d}$$

$$H^{\text{loc sing log}}_{\text{3PN 3D reg}} = 0, \tag{54a}$$

$$\begin{aligned}\Delta H_{\text{3PN}} &= \lim_{d\to 3} H^{\text{loc sing log}}_{\text{3PN}}(d) \\ &= \lim_{d\to 3} \sum_{A=1}^{10} H^{\text{loc sing log}}_{A}(d) + (1 \leftrightarrow 2) \\ &= 32 \frac{G^3\, m_1\, m_2}{r_{12}^3} \left(\left[\mathbf{p}_1^2 - 3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\right] + (1 \leftrightarrow 2) \right) \\ &\quad \times \sum_{A=1}^{10} c'_A(3).\end{aligned} \tag{54b}$$

From Eq. (54b) one determines the values of both ambiguity parameters:

$$\omega^{\text{dim reg}}_{\text{kinetic}} = \frac{41}{24}, \tag{55a}$$

$$\omega^{\text{dim reg}}_{\text{static}} = 0\,. \tag{55b}$$

Conformally flat truncations of general relativity

$$h_{ij}^{\mathsf{TT}} = 0$$

$$\Downarrow$$

- Violation of global Poincaré invariance (starting at 2PN order).

- No cancellation of poles leading to a formally infinite 3PN Hamiltonian.