# On a solution in Einstein-Maxwell-dilaton gravity

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**Abstract:** The existence of Einstein-Maxwell-dilaton (EMD) plus fluid system for the case of stationary cylindrically symmetric spacetimes is considered. The exact inhomogeneous $\varepsilon$-order solution is found, where the parameter $\varepsilon$ parametrizes the non-minimally coupled electromagnetic field. It is shown that the found solution also applies in the string-inspired charged gravity framework.

## 1 Preliminaries

- We search for cylindrically symmetric stationary spacetimes with existing local coordinates $(x^0, x^1, x^2, x^3) = (t, \varphi, z, r)$ adapted to Killing fields $\partial_t,\ \partial_\varphi,\ \partial_z$.
- The metric tensor is written $g = \eta_{\mu\nu}\Theta^{\hat{\mu}} \otimes \Theta^{\hat{\nu}}$, where $(\eta_{\mu\nu}) = \mathrm{diag}(\ 1,\ -1,\ -1,\ -1)$, and local coframe fields $\Theta^{\hat{\mu}}$ are defined by
$$\begin{aligned} \Theta^{\hat{0}} &= \mathrm{e}^{\alpha}(\,\mathrm{d}t + f\,\mathrm{d}\varphi)\ , & \Theta^{\hat{1}} &= l\,\mathrm{d}\varphi\ , \\ \Theta^{\hat{2}} &= \mathrm{d}z\ , & \Theta^{\hat{3}} &= \mathrm{e}^{\delta}\,\mathrm{d}r\ , \end{aligned} \tag{1}$$
with $f,\ l,\ \alpha,\ \delta$ being functions of $r$ only.
- The spacetime is filled with a charged perfect fluid, characterised by the pressure $p(r)$ and the energy density $\mu(r)$, and with a massless scalar field – dilaton $\phi(z, r)$.
- The $\varepsilon$–order EMD action is given by
$$\boxed{S = \int\limits_{\mathcal{M}} \Big[ *R + 16\pi * \mu - \mathrm{d}\phi \wedge * \mathrm{d}\phi + 2\varepsilon\ \mathrm{e}^{\phi}\ F \wedge *F \Big]\ , \qquad (2)}$$

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where $\varepsilon$ is a real parameter, $R$ is the Ricci scalar of the metric tensor, $F$ is the electromagnetic field 2-form.

- We claim to obtain a EMD solution which is of the first order in the parameter $\varepsilon$. One can decompose $\phi$ into the sum of zero-order solution $\phi^{(0)}$ and a $\varepsilon$-order correction $\phi^{(1)}$ as

$$\phi = \phi^{(0)} + \varepsilon\phi^{(1)} \ . \tag{3}$$

## 2 Zeroth-order solution

- The scalar field equation of motion $* \mathrm{d} * \mathrm{d}\, \phi = 0$ is integrated to

$$\phi = \phi_0 + \phi_1 z + \phi_2 \int l^{-1} \mathrm{e}^{\delta-\alpha} \ \mathrm{d}r \, , \quad \phi_1\phi_2 = 0 \ , \tag{4}$$

with constants $\phi_0$, $\phi_1$ and $\phi_2$.

- Bianchi identity

$$u \wedge *[\mu \ \mathrm{d}u + \ \mathrm{d}(p\ u)] = 0 \tag{5}$$

reduce the problem to solving five equations for six unknowns.

- One obtains a system of two non–linear second–order equations. In particular case when $\phi_2 = 0$ the equations (3) and (4) immediately imply

$$\phi^{(0)} = \phi_0 + \phi_1 z \ . \tag{6}$$

This dilaton is common in papers [2, 3, 4]. Dilaton blows up at the $z$-infinities, which can be condidered as additional sources of scalar charge.

- The solution of the Einstein equations can be written in the form

$$\begin{aligned} \mathrm{d}s^2 \ &= \ \mathrm{e}^{2\alpha} \left( \mathrm{d}t + f\, \mathrm{d}\varphi\right)^2 - l^2\, \mathrm{d}\varphi^2 \\ &- \ \ \mathrm{d}z^2 - C^{-2} l^{-2} \left( \mathrm{d}\mathrm{e}^{\alpha}\right)^2 \ \ , \end{aligned} \tag{7}$$

the metric functions $f$ and $l^2$ being given by

$$\begin{aligned} f &= -\frac{\Omega}{C}\mathrm{e}^{-2\alpha} + F \ , \\ C^2 l^2 &= \Omega^2 \mathrm{e}^{-2\alpha} + \frac{1}{4}\phi_1^2 \mathrm{e}^{2\alpha} + D\alpha + E \ , \end{aligned} \tag{8}$$

with $\Omega,\ C,\ D,\ E,\ F$ as integration constants. The formula (8) is expressed in terms of an arbitrary non-constant $C^2$ function $\alpha$ that reflects the radial coordinate rescaling possibility.

## 3 First-order solution

- A charge is distributed with a current density $j(r,z)$ through a spacetime. Consider the case when the Lorentz force acting on the fluid particles vanishes. In the fluid rest frame only a magnetic field survives. Excluding the spacetimes with currents parallel to the axis the electromagnetic field 2-form is given by

$$F = \mathrm{B}_{\hat{r}}\ \Theta^{\hat{1}} \wedge \Theta^{\hat{2}} + \mathrm{B}_{\hat{z}}\ \Theta^{\hat{3}} \wedge \Theta^{\hat{1}} \ . \tag{9}$$

- Generalized Maxwell equations are

$$- * \ \mathrm{d} * (\mathrm{e}^{\phi^{(0)}}\ F) = \frac{4\pi}{\varepsilon}\ j \ . \tag{10}$$

- Since we neglect terms of $\mathcal{O}(\varepsilon^2)$, we have arrived to the following conditions for the electromagnetic field

$$\begin{aligned} *(u \wedge F) \wedge \Theta^{\hat{2}} &= 0 \ , \\ u \wedge *F &= 0 \ . \end{aligned} \tag{11}$$

The conditions (11) are solved by a purely longitudinal magnetic field, parallel to the rotation axis,

$$\mathrm{B}_{\hat{z}} = B\mathrm{e}^{-\frac{1}{2}\phi^{(0)} - \alpha} \ , \qquad B = \mathrm{const} \ . \tag{12}$$

Fluid particles are the charge carriers, $j = \rho u$.

- The exterior derivative of (12) does not vanish which means that we deal with a current of the magnetic monopoles. One has to introduce a magnetic charges current density 1-form $j_m$ by

$$\frac{4\pi}{\varepsilon}\, j_m = *\,\mathrm{d}F = -\frac{1}{2}B\phi_1 \mathrm{e}^{-\frac{1}{2}\phi^{(0)}-\alpha}\Theta^{\hat{0}}\ . \tag{13}$$

- The dilaton according to the (4) becomes equal to

$$\phi = \phi_0 + \tilde{\phi}_1 z\ ,$$

where $\tilde{\phi}_1$ is constant including $\varepsilon$–order correction.

- The Bianchi identity in our case, provided that the scalar field equation of motion (4) is fulfilled, is

$$u \wedge *[\mu\ \mathrm{d}u + \mathrm{d}(p\ u) + \varepsilon\ \rho\ \mathrm{e}^{\phi^{(0)}}\ F] = 0\ . \tag{14}$$

- One obtains the metric (7) with the following functions

$$\boxed{\begin{aligned} f &= -\frac{\Omega}{C}\mathrm{e}^{-2\alpha} + F\ , \\ C^2l^2 &= \Omega^2\mathrm{e}^{-2\alpha} + \frac{1}{4}\tilde{\phi}_1^2\mathrm{e}^{2\alpha} - 2\varepsilon B^2\alpha^2 + D\alpha + E. \end{aligned}} \tag{15}$$

- The energy density, the pressure and the charge density are

$$\boxed{\begin{aligned} 16\pi\mu &= \left[D + 2\varepsilon B^2\left(1 - 2\alpha\right)\right]\mathrm{e}^{-2\alpha} - \tilde{\phi}_1^2\ , \\ 16\pi p &= \left[D - 2\varepsilon B^2\left(1 + 2\alpha\right)\right]\mathrm{e}^{-2\alpha} + \tilde{\phi}_1^2\ , \\ 2\pi\rho &= -\varepsilon\Omega B\mathrm{e}^{\frac{1}{2}\phi^{(0)}-3\alpha}\ . \end{aligned}} \tag{16}$$

## 4 Case with more scalar fields

- It is straightforward to generalize the zeroth order solution (8) and the first order solution (15) in the case when more scalar fields are present. The motivation comes from string theory.

  We shall consider $N$ massless scalar fields $\phi_i$, $i = 1, 2, ..N$, and $N$ non-minimally coupled massless scalar fields $\psi_i$. The total action (2) can be rewritten as

$$S = \int_{\mathcal{M}} \Big[ *R + 16\pi * \mu + 2\varepsilon\, \mathrm{e}^{\phi}\; F \wedge *F \\ - \sum_i (\,\mathrm{d}\phi_i \wedge * \,\mathrm{d}\phi_i + \mathrm{e}^{-2\phi_i}\, \mathrm{d}\psi_i \wedge * \,\mathrm{d}\psi_i)\Big] \; . \tag{17}$$

## 4.1 Zeroth order in $\varepsilon$

- The Klein-Gordon equation for each of the scalar fields $\phi_i$ still holds, while the equations of motion for the scalar fields $\psi_i$ are given by

$$* \,\mathrm{d}(\mathrm{e}^{-2\phi_i} * \,\mathrm{d}\psi_i) = 0 \; . \tag{18}$$

- Provided that neither $\phi_i$ nor $\psi_i$ depends on the radial coordinate one has

$$\phi_i = \phi_{i0} + \phi_{i1} z \; , \;\; \psi_i = \psi_{i0} + \mathrm{e}^{\phi_i} \psi_{i1} \; , \tag{19}$$

  which inserted into the (18) yields $\phi_{i1}\psi_{i1} = 0$.
  In (19) $\phi_{i0}, \phi_{i1}, \psi_{i0}, \psi_{i1}$ are integration constants.

- Thus the only non-trivial zero-order solution is given by

$$\phi_i^{(0)} = \phi_{i0} + \phi_{i1} z \; , \;\; \psi_i^{(0)} = \psi_{i0} \; . \tag{20}$$

  The solution (8) remains unaffected provided $\phi_1^2$ is replaced by $\Phi_1^2 = \Sigma\, \phi_{i1}^2$.

## 4.2 First order in $\varepsilon$

- We continue to keep our *Ansatz* (12) for the electromagnetic field. It also means that scalar fields $\phi_i$ are given by

$$\phi_i = \phi_{i0} + \tilde{\phi}_{i1} z \; , \tag{21}$$

  with constants $\tilde{\phi}_{i1}$ including the $\varepsilon$-order corrections to $\phi_{i1}$.

- The resulting metric and physical quantities are still given by (15) and (16) provided $\tilde{\phi}_1^2$ is replaced by $\tilde{\Phi}_1^2 = \Sigma\, \tilde{\phi}_{i1}^2$.

- Solving the equations of motion (18) for the fields $\psi_i$ one obtains

$$\psi_i = \psi_{i0} + \varepsilon \mathrm{e}^{\phi_{i1} z}[A_i \cos(v_i z) + B_i \sin(v_i z)]\ \eta_i(r)\ , \tag{22}$$

where $A_i,\ B_i$ and $v_i$ are arbitrary constants and the functions $\eta_i$ are solutions of second-order equations that can be transformed into the form

$$\mathrm{e}^{-2\alpha} \frac{\mathrm{d}}{\mathrm{d}m} \left( l^2 \frac{\mathrm{d}\eta_i}{\mathrm{d}m} \right) - \left(\phi_{i1}^2 + v_i^2\right)\ \eta_i = 0\ . \tag{23}$$

## 5 String-inspired theory of gravity

- The total string-inspired effective action (2) can be rewritten as [3, 5]

$$\boxed{\begin{aligned} S_{\text{eff}} = \int\limits_{\mathcal{M}} \Big[ &*R + 16\pi * \mu + 2\varepsilon\ \mathrm{e}^{\phi}\ F \wedge *F - 4\varepsilon\ \psi\ F \wedge F \\ &- \sum_i (\ \mathrm{d}\phi_i \wedge * \mathrm{d}\phi_i + \mathrm{e}^{-2\phi_i}\ \mathrm{d}\psi_i \wedge * \mathrm{d}\psi_i) \\ &- 8\pi^2 \varepsilon \left( \mathrm{e}^{\phi} e(\mathcal{M}) + 4\psi p_1(\mathcal{M}) \right) \Big]\ . \end{aligned} \quad (24)}$$

In terms of the inverse string tension $\alpha'$ (Regge slope) and the string coupling constant g the parameter $\varepsilon$ is expressed like $\varepsilon = \frac{\alpha'}{4\mathrm{g}^2}$. The Euler class $e(\mathcal{M})$ and the first Pontrjagin class $p_1(\mathcal{M})$ of $T\mathcal{M}$ in (24) are in four dimensions equal to

$$e(\mathcal{M}) = \frac{1}{8\pi^2} \left( R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 4 R_{\alpha\beta} R^{\alpha\beta} + R^2 \right) \eta \tag{25}$$

and

$$p_1(\mathcal{M}) = -\frac{1}{8\pi^2} \mathrm{Tr}\ \Omega \wedge \Omega = -\frac{1}{32\pi^2} R^{\alpha\beta}{}_{\gamma\delta} R^{\sigma\tau}{}_{\alpha\beta}\ \eta^{\gamma\delta}{}_{\sigma\tau}\ \eta\ , \tag{26}$$

$\eta$ is the volume element with components $\eta_{\alpha\beta\gamma\delta}$. The fields $\phi_N \equiv \phi$ and $\psi_N \equiv \psi$ may be referred to as the dilaton and axion respectively.

- We keep an extra contribution arising from a field $\mu$. Physically it may represent an energy density of a fermionic matter, that is in our model approximated by a perfect fluid of a pressure $p$.
- For the basis (1) it turns out that the term as well the Euler class (25) as well as the Pontrjagin class (26) are identically vanishing.
- Since the magnetic field (12) is purely longitudinal, our problem reduces to the Einstein-Maxwell-dilaton plus fluid system.
- This completes the proof that the metric (15) after appropriate replacement $\tilde{\phi}_1^2 \to \tilde{\Phi}_1^2 = \Sigma\,\tilde{\phi}_{i1}^2$ constitutes string-inspired solution. The scalar fields are given by (21) and (22), subject to the equation (23).

**References**


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