A wideband acoustic GW detector: Dual Sphere

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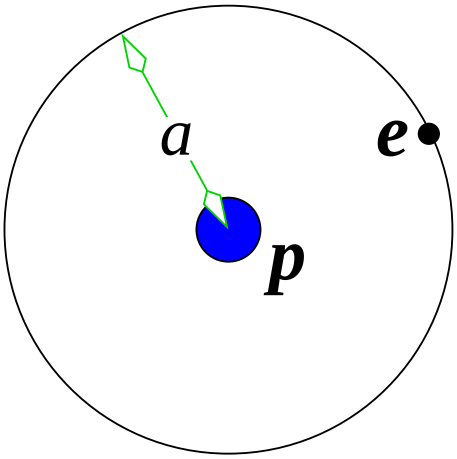
Preamble

Gravitation is, compared to other fundamental forces, ***extremely weak***. Take e.g. a proton and an electron in a hydrogen atom:



$$F_{\mathrm{grav}} = \boldsymbol{G}\,\frac{m_{\mathrm{e}} m_{\mathrm{p}}}{a^2} \qquad F_{\mathrm{coul}} = \frac{1}{\boldsymbol{4\pi\varepsilon_0}}\,\frac{q_{\mathrm{e}} q_{\mathrm{p}}}{a^2}$$

$$\frac{F_{\mathrm{grav}}}{F_{\mathrm{coul}}} \simeq 4 \times 10^{-41}$$

GWs are therefore also very weak. ***Quadrupole formula*** yields order of magnitude estimate of GW amplitude:

$$h \sim \frac{R_{\mathrm{Schw}}}{r}\,\frac{v^2}{c^2}$$

where:

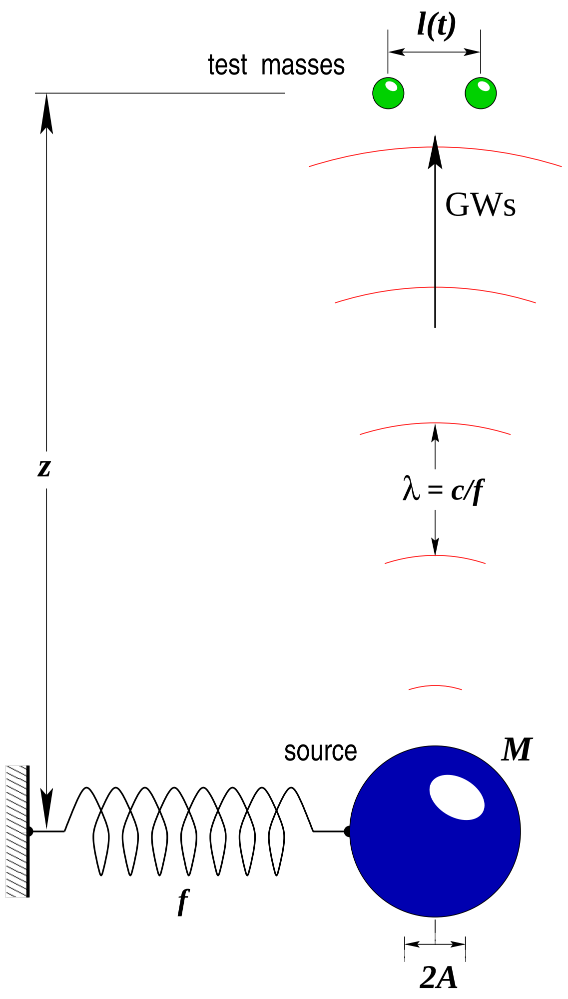
— R_{Schw} = source's gravitational radius

— r = distance to source

— v = typical internal velocity

— c = speed of light

Earth scale example:



Take e.g.

$$A = 0.1 \text{ metres} \ , \quad f = 1 \text{ kHz} \ , \quad M = 1 \text{ ton}$$

to obtain

$$h_+(t) = \frac{2\,\delta l}{l} \simeq 10^{-39} \left(\frac{1500 \text{ km}}{z}\right) \ , \quad h_\times(t) = 0$$

System looses energy to ***gravitational bremsstrahlung***, but very little:

$$\frac{\delta A}{A} \simeq 3 \times 10^{-17} \ , \quad \text{during whole age of Universe!}$$

Several consequences follow:

1. Most intense sources are ***astrophysical/csomological***
2. They are very far, and $h \sim 10^{-18} - 10^{-24}$ on Earth
3. GWs are in practice ***plane*** on arrival to antenna

There are two essential issues to GW detection:

a) ***Interaction GW-detector***

b) ***Level of noise in detector*** (sensitivity)

Plan of this talk

To address these issues, for a ***spherical detector***

Brief review of Newman–Penrose formulas

In a four–dimensional, Riemannian space–time manifold with a *metric* field $g_{\mu\nu}(x)$, a *vector tetrad* can be defined pointwise,

$$\boldsymbol{e}_{(a)} = e^{\mu}_{(a)}(x)\,\frac{\partial}{\partial x^{\mu}}\ , \qquad a = 1, \ldots, 4$$

or

$$\boldsymbol{e}_{(1)} \equiv \boldsymbol{l}\ , \quad \boldsymbol{e}_{(2)} \equiv \boldsymbol{n}\ , \quad \boldsymbol{e}_{(3)} \equiv \boldsymbol{m}\ , \quad \boldsymbol{e}_{(4)} \equiv \overline{\boldsymbol{m}}$$

with *constant* scalar products:

$$\boldsymbol{l} \cdot \boldsymbol{n} = -\boldsymbol{m} \cdot \overline{\boldsymbol{m}} = -1\ \ , \qquad \text{all others} = 0$$

i.e.,

$$\boldsymbol{e}_a \cdot \boldsymbol{e}_b \equiv \eta_{(a)(b)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 \end{pmatrix}$$

This is rightly called a *null tetrad*.

The *Riemann tensor* can be split up as

$$R_{(a)(b)(c)(d)} = C_{(a)(b)(c)(d)} + E_{(a)(b)(c)(d)} + G_{(a)(b)(c)(d)}$$

where

$$\begin{aligned} C_{(a)(b)(c)(d)} &= \text{Weyl tensor} \\ E_{(a)(b)(c)(d)} &= \tfrac{1}{2}\left[\eta_{(a)(c)}R_{(b)(d)} - \eta_{(a)(d)}R_{(b)(c)} + (\eta \leftrightarrow R)\right] \\ G_{(a)(b)(c)(d)} &= -\tfrac{1}{6}\left(\eta_{(a)(c)}\eta_{(b)(d)} - \eta_{(a)(d)}\eta_{(b)(c)}\right) R \end{aligned}$$

and

$$R_{(a)(c)} \equiv \eta^{(b)(d)}\, R_{(a)(b)(c)(d)} \ , \qquad R \equiv \eta^{(a)(c)}\, R_{(a)(c)}$$

are the *Ricci tensor* and the *curvature scalar*.

The **20** independent components of the Riemann tensor are expediently expressed by means of **10** *complex* scalar funcions, to wit,

- **Weyl**:

$$\begin{aligned} \Psi_0 &\equiv C_{(1)(3)(3)(1)} \\ \Psi_1 &\equiv C_{(1)(2)(3)(1)} \\ \Psi_2 &\equiv C_{(1)(3)(2)(4)} \\ \Psi_3 &\equiv C_{(1)(2)(2)(4)} \\ \Psi_4 &\equiv C_{(2)(4)(4)(2)} \end{aligned}$$

- **Ricci**:

$$\Phi \equiv \begin{pmatrix} \Phi_{00} & \Phi_{01} & \Phi_{02} \\ \Phi_{10} & \Phi_{11} & \Phi_{12} \\ \Phi_{20} & \Phi_{21} & \Phi_{22} \end{pmatrix} \equiv -\frac{1}{2} \begin{pmatrix} R_{(1)(1)} & R_{(1)(3)} & R_{(3)(3)} \\ R_{(1)(4)} & \frac{1}{2}\left[R_{(1)(2)} + R_{(3)(4)}\right] & R_{(2)(3)} \\ R_{(4)(4)} & R_{(2)(4)} & R_{(2)(2)} \end{pmatrix}$$

- **Curvature scalar:**

$$\Lambda \equiv \frac{1}{2}\, R \equiv \frac{1}{2}\left[R_{(1)(2)} - R_{(3)(4)}\right]$$

where e.g., $C_{(1)(2)(3)(1)} \equiv C_{\mu\nu\rho\sigma}\, l^\mu n^\nu m^\rho l^\sigma$, $R_{(1)(3)} \equiv R_{\mu\nu}\, l^\mu m^\nu$, etc. Note that Φ is a *Hermitian matrix*. All these quantities have suitable transformation properties under the action of tetrad changes preserving the scalar products (), which are $SL(2, \boldsymbol{C})$ transformations.

Algebraic structure of a *metric* GW

A *metric* theory of gravity is one which conforms to the *Strong Equivalence Principle*. General Relativity is one such theory, but not the only one —think e.g., of Brans–Dicke.

We consider *plane* waves, as they will be observed *far* from their source. They are characterised by a Riemann tensor, the only *observable* magnitude, which is a function of a *sole* null variable v. Also, a GW will be assumed a *vacuum field* —a ***theory dependent*** concept, this one.

The algebraic properties of $R_{\mu\nu\rho\sigma}$, together with the differential *Bianchi identities* $R_{\mu\nu[\rho\sigma;\lambda]}=0$, imply that only *six* (out of *twenty*) components of the Riemann tensor are not identically zero. If the null variable upon which this tensor depends be chosen to trace the null ray which is tangent to the tetrad vector $\boldsymbol{l}$ then these components are the four scalars

$$\Psi_2(v), \Psi_3(v), \Psi_4(v) \;\; \text{and} \;\; \Phi_{22}(v)$$

In a local Lorentz frame one can take

$$\boldsymbol{l} = \frac{1}{\sqrt{2}}\,(\boldsymbol{e}_t - \boldsymbol{e}_z)\ , \qquad \boldsymbol{n} = \frac{1}{\sqrt{2}}\,(\boldsymbol{e}_t + \boldsymbol{e}_z)$$
$$\boldsymbol{m} = \frac{1}{\sqrt{2}}\,(\boldsymbol{e}_x + i\boldsymbol{e}_y)\ , \qquad \overline{\boldsymbol{m}} = \frac{1}{\sqrt{2}}\,(\boldsymbol{e}_x - i\boldsymbol{e}_y)$$

so that the scalars () read

$$\begin{aligned}
\Psi_2 &= -\frac{1}{6}\,R_{0z0z} \\
\Psi_3 &= -\frac{1}{2}\,(-R_{0x0z} + iR_{0y0z}) \\
\Psi_4 &= -R_{0x0x} + R_{0y0y} + 2iR_{0x0y} \\
\Phi_{22} &= -R_{0x0x} - R_{0y0y}
\end{aligned}$$

Therefore only the so called ***electric*** components of the Riemann tensor, R_{0i0j}, are relevant for GW detection.

A given *theory of gravitation* makes its own characteristic predictions as to the nature of GWs, i.e., which excitations do actually happen in practice. *All* possibilities can be classified in terms of the scalars () which can be made to vanish by those $SL(2, \boldsymbol{C})$ transformations leaving invariant the wave propagaton vector, $\boldsymbol{l}$. There are *six* possibilities (Lee *et al., PRD* **8**, 3308 (1973)):

$$\begin{array}{lll}
II_6: & & \Psi_2 \not\equiv 0 \\
III_5: & \Psi_2 \equiv 0 \ , & \Psi_3 \not\equiv 0 \\
N_3: & \Psi_2 \equiv \Psi_3 \equiv 0 \ , & \Psi_4, \Phi_{22} \not\equiv 0 \\
N_2: & \Psi_2 \equiv \Psi_3 \equiv \Phi_{22} \equiv 0 \ , & \Psi_4 \not\equiv 0 \\
O_1: & \Psi_2 \equiv \Psi_3 \equiv \Psi_4 \equiv 0 \ , & \Phi_{22} \not\equiv 0 \\
O_0: & \Psi_2 \equiv \Psi_3 \equiv \Psi_4 \equiv \Phi_{22} \equiv 0 &
\end{array}$$

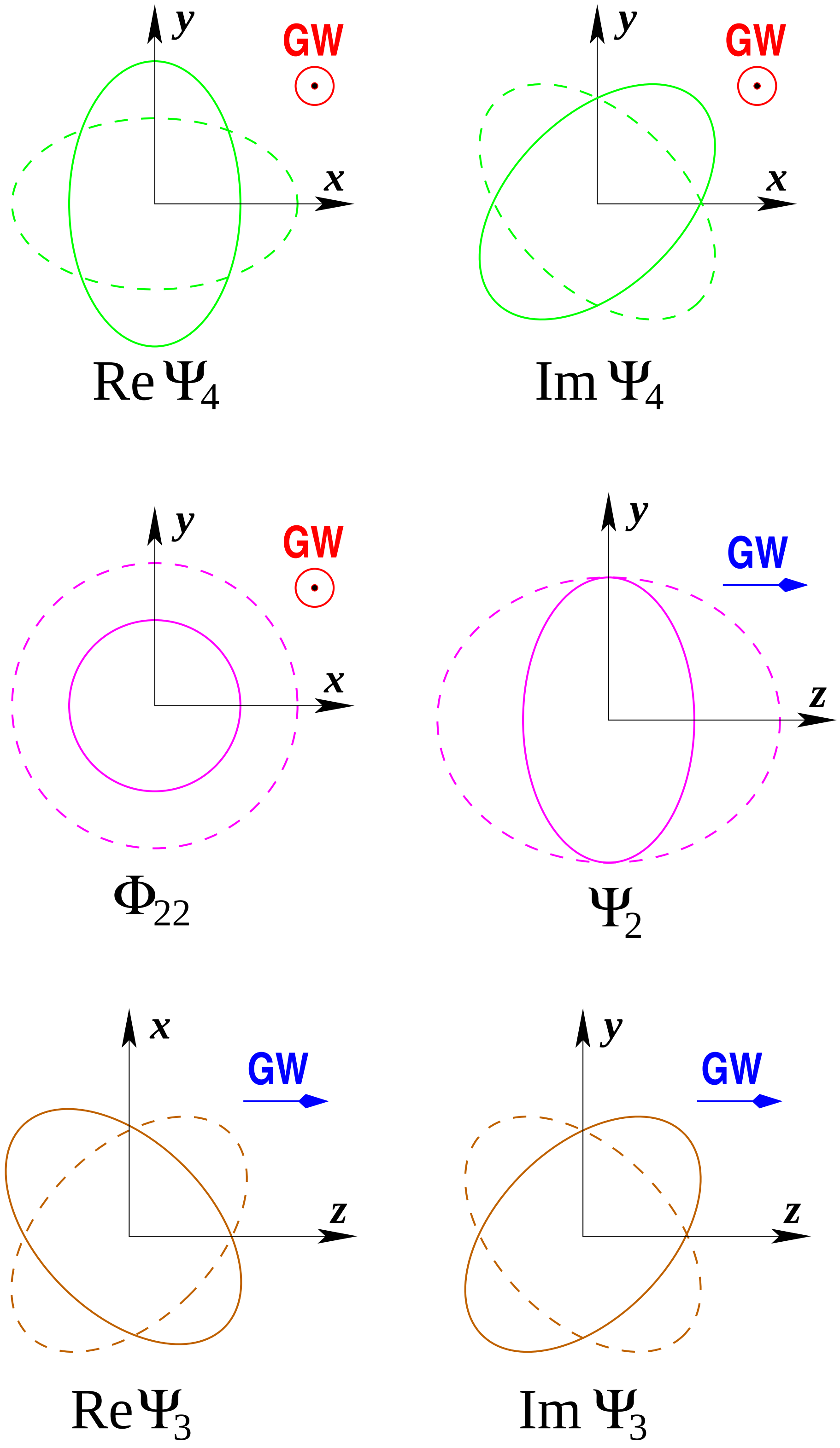
This is a hierarchical classification, from more to less general GWs. Most well known cases are

General Relativity:	Type N_2
Brans–Dicke:	Type N_3

Other cases:

No gravity waves:	Type O_0
Purely scalar waves:	Type O_1

GW Amplitudes cause *tidal* displacements on a set of test particles, originally on a spherical surface, according to the scheme



Physical Principle of the acoustic GW detector

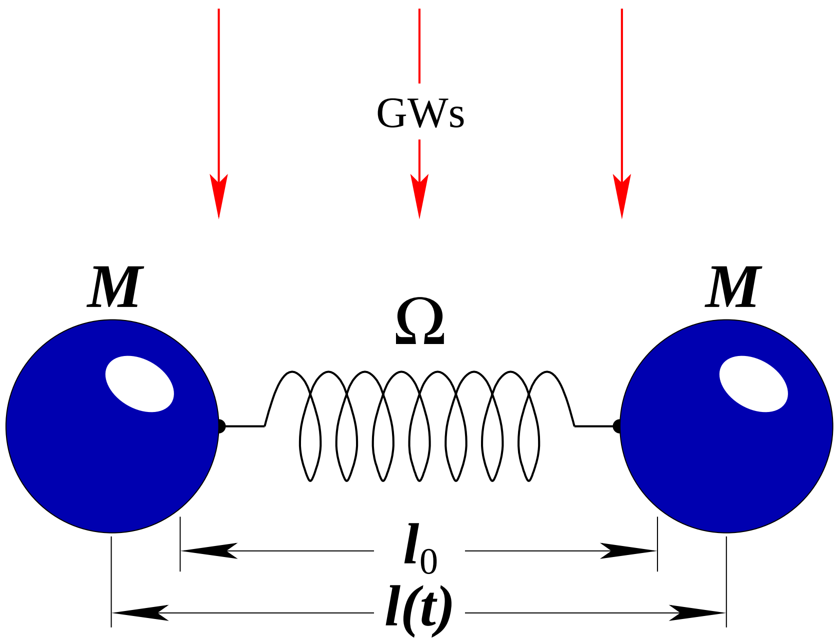
Two otherwise ***free masses*** feel the presence of a GW through ***tides***, or *geodesic deviations*, or relative accelerations.

For ***weak signals*** the equation of motion for the relative separation is approximately given by

$$\ddot{\ell}(t) = \frac{1}{2}\,\ddot{h}(t)\,\ell(t)$$

If the two masses are ***linked with a spring*** of characteristic frequency Ω then the above changes into

$$\ddot{\ell}(t) + \Omega^2\;[\ell(t) - \ell_0] = \frac{1}{2}\,\ddot{h}(t)\,\ell(t)$$



Thus acoustic dectors ***selectively amplify*** GW signal at frequency $\mathbf{\Omega}$.

Mathematics of the acoustic GW detector

The basic assumption of the mathematical model is that the gravitational antenna behaves like an *elastic isotropic solid*, whose dynamics is assessed in terms of a *non-relativistic* theory of Elasticity, e.g., the classical theory of Poisson, Boussinesq, *et al.*, presented in standard textbooks such as Landau & Lifshitz (Pergamon 1970). According to such theory, the field $\mathbf{u}(\mathbf{x},t)$ of the solid's displacements, relative to their undisturbed positions, are governed by the partial differential equations

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} - \mu\nabla^2\mathbf{u} - (\lambda+\mu)\,\nabla(\nabla\!\cdot\!\mathbf{u}) = \mathbf{f}(\mathbf{x},t)$$

when a *density of force* $\mathbf{f}(\mathbf{x},t)$ acts upon the solid. Here ρ is the solid's density, and λ and μ are its elastic Lamé coefficients —related to the Young modulus and Poisson ratio.

Equations () are *linear*, and so a wealth of techniques to solve them are available in the literature. We choose a **Green function** expansion method. Attention will be restricted to *separable* driving forces of the form

$$\mathbf{f}(\mathbf{x},t) = \mathbf{f}(\mathbf{x})\,g(t)$$

or linear combinations thereof which, as we shall see, suffice to our purposes. The *formal* solution is written, in this case,

$$\mathbf{u}(\mathbf{x},t) = \int_0^\infty \mathbf{S}(\mathbf{x};t-t')\,g(t')\,dt'$$

where $\mathbf{S}(\mathbf{x},t)$ is the Green function, easily calculated:

$$\mathbf{S}(\mathbf{x};t) = \begin{cases} 0 & \text{if} \quad t\leq 0 \\ \sum_N \frac{f_N}{\omega_N}\,\mathbf{u}_N(\mathbf{x})\sin\omega_N t & \text{if} \quad t\geq 0 \end{cases}$$

Here,

$$f_N \equiv \frac{1}{M} \int_{\text{Solid}} \mathbf{u}_N^*(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})\, d^3x$$

and $\mathbf{u}_N(\mathbf{x})$ are *eigen–solutions* to

$$\mu\nabla^2\mathbf{u}_N + (\lambda+\mu)\,\nabla(\nabla\cdot\mathbf{u}_N) = -\omega_N^2\rho\mathbf{u}_N$$

with suitable *boundary conditions.* These functions are *normalised* according to

$$\int_{\text{Solid}} \mathbf{u}_{N'}^*(\mathbf{x}) \cdot \mathbf{u}_N(\mathbf{x})\,\rho(\mathbf{x})\, d^3x = M\,\delta_{N'N}$$

where M is the total mass of the solid, and an asterisk denotes complex conjugation. So, explicitely,

$$\mathbf{u}(\mathbf{x},t) = \sum_N \frac{f_N}{\omega_N}\,\mathbf{u}_N(\mathbf{x})\,g_N(t)$$

with the abbreviated notation

$$g_N(t) \equiv \int_0^t g(t')\,\sin\omega_N(t-t')\,dt'$$

One thing to stress is that, in all the above formulæ, the index N will generally be a *multiple index*, e.g.,

$$N = \{nlm\}$$

The GW *tidal* forces

It will be assumed that GW show as *tidal forces*, so that

$$f_i(\mathbf{x},t) = \rho c^2\, R_{0i0j}(t)\, x_j$$

where $R_{0i0j}(t)$ is the GW's Riemann tensor *calculated at the centre of the elastic body*. If quadratic terms in the metric perturbations $h_{\mu\nu}$ are neglected then

$$R_{0i0j} = \frac{1}{2}\,(h_{ij,00} - h_{0i,0j} - h_{0j,0i} + h_{00,ij})$$

As we see, the tidal force density () is of the *separable* type, just described. We shall adopt the following decomposition:

$$\mathbf{f}(\mathbf{x},t) = \mathbf{f}^{(0,0)}(\mathbf{x})\, g^{(0,0)}(t) \ + \sum_{m=-2}^{2} \mathbf{f}^{(2,m)}(\mathbf{x})\, g^{(2,m)}(t)$$

with

$$f_i^{(0,0)}(\mathbf{x}) = \rho\, E_{ij}^{(0,0)}\, x_j \quad , \quad g^{(0,0)}(t) = \frac{4\pi}{3}\, E_{ij}^{*(0,0)}\, R_{0i0j}(t)\, c^2$$

$$f_i^{(2,m)}(\mathbf{x}) = \rho\, E_{ij}^{(2,m)}\, x_j \quad , \quad g^{(2,m)}(t) = \frac{8\pi}{15}\, E_{ij}^{*(2,m)}\, R_{0i0j}(t)\, c^2$$

The possibility of making this decomposition depends on the symmetric nature of the Riemann tensor's "electric" components R_{0i0j}. Here the $E_{ij}^{(l,m)}$ are a set of six symmetric 3-tensors, in terms of which any other such tensor can be expressed by suitable linear combination. These tensors $E_{ij}^{(l,m)}$ are associated to the monopole and quadrupole irreducible parts of a general symmetric tensor. They are chosen to verify

$$E_{ij}^{(l,m)}\, n_i\, n_j = Y_{lm}(\theta,\varphi)$$

where n_i are the Cartesian components of the unit vector. A suitable *representation* for them is

$$E_{ij}^{(0,0)} = \left(\frac{1}{4\pi}\right)^{\frac{1}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_{ij}^{(2,0)} = \left(\frac{5}{16\pi}\right)^{\frac{1}{2}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \ , \quad E_{ij}^{(2,\pm 1)} = \left(\frac{15}{32\pi}\right)^{\frac{1}{2}} \begin{pmatrix} 0 & 0 & \mp 1 \\ 0 & 0 & -i \\ \mp 1 & -i & 0 \end{pmatrix}$$

$$E_{ij}^{(2,\pm 2)} = \left(\frac{15}{32\pi}\right)^{\frac{1}{2}} \begin{pmatrix} 1 & \pm i & 0 \\ \pm i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

There are *two* important facts about the decomposition (), (5) and (5):

- The time factors $g^{(l,m)}(t)$ carry the *dynamics* of the incoming GW, while the space factors $\mathbf{f}^{(l,m)}(\mathbf{x})$ are pure *form factors*, independent of both the detector's shape and the driving GW.
- It is *independent* of the theory of gravitation which governs in fact the behaviour of GWs —provided it is a *metric* theory, i.e., conforming to the Equivalence Principle.

The solution to the problem for a general metric GW **and** detector shape is thus

$$\mathbf{u}(\mathbf{x},t) = \sum_N \omega_N^{-1} \mathbf{u}_N(\mathbf{x}) \left[\sum_{\substack{l\,=\,0 \text{ and } 2 \\ m\,=\,-l,\ldots,l}} f_N^{(l,m)}\, g_N^{(l,m)}(t) \right]$$

The spherical antenna

It has *two* families of eigenmodes:

$$\mathbf{u}^P_{nlm}(\mathbf{x})\,e^{\pm i\omega^T_{nl}t} \qquad , \qquad \mathbf{u}^T_{nlm}(\mathbf{x})\,e^{\pm i\omega^P_{nl}t}$$

called *spherioidal* and *toroidal*, respectively. *Only the former couple to GWs*, so superindex P is dropped. It is found that

$$\mathbf{u}(\mathbf{x},t) = \sum_{n=1}^{\infty}\frac{a_{n0}}{\omega_{n0}}\,\mathbf{u}_{n00}(\mathbf{x})\,g^{(0,0)}_{n0}(t) \;+\; \sum_{n=1}^{\infty}\frac{a_{n2}}{\omega_{n2}}\left[\sum_{m=-2}^{2}\mathbf{u}_{n2m}(\mathbf{x})\,g^{(2,m)}_{n2}(t)\right]$$

$$g^{(l,m)}_{nl}(t) \equiv \int_0^t g^{(l,m)}(t')\,\sin\omega_{nl}(t-t')\,dt'$$

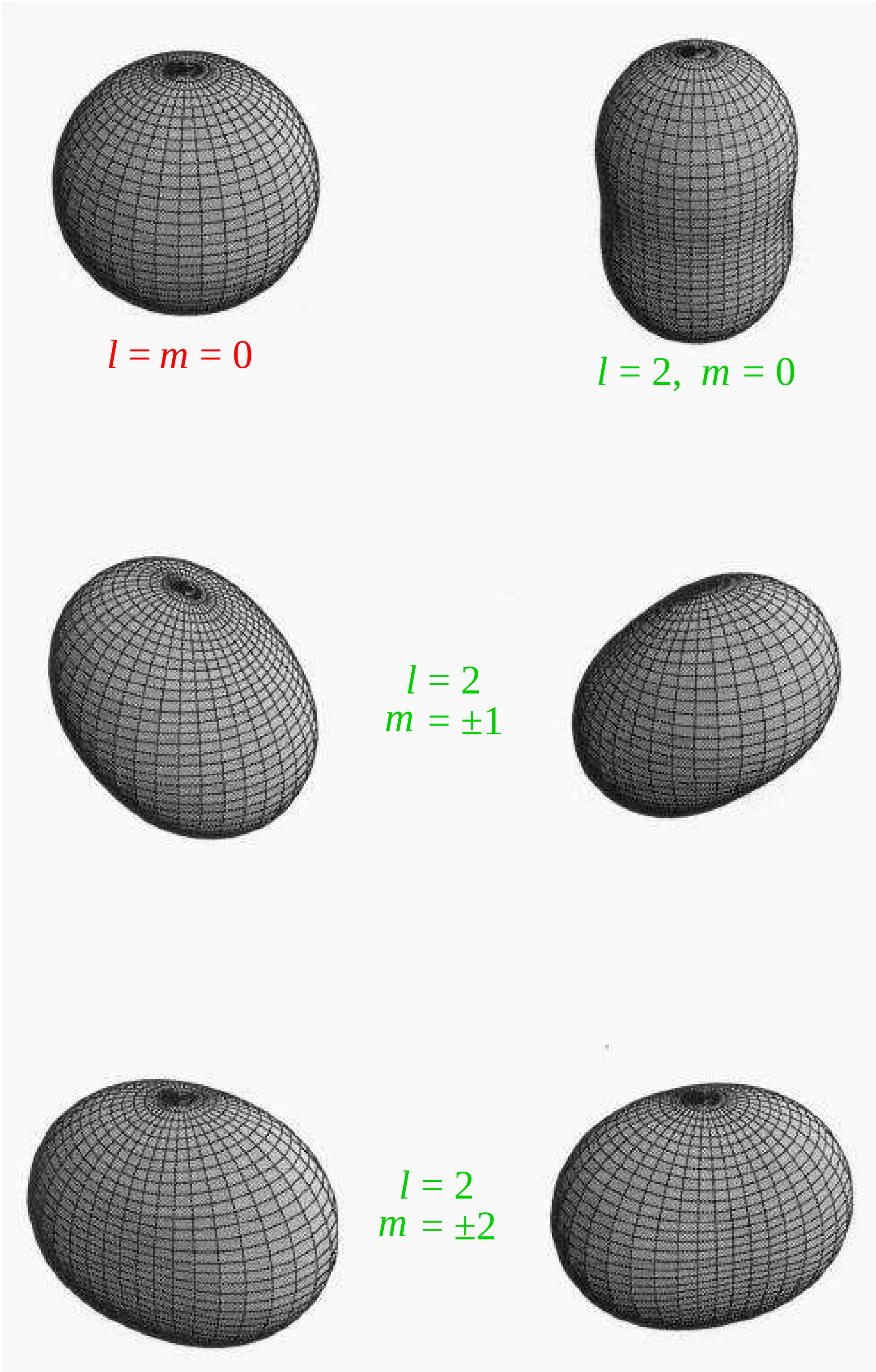
where

$$\begin{aligned} a_{n0} &= -\frac{1}{M}\int_0^R A_{n0}(r)\,\rho\,r^3\,dr \\ a_{n2} &= -\frac{1}{M}\int_0^R \left[A_{n2}(r) + 3\,B_{n2}(r)\right]\,\rho\,r^3\,dr \end{aligned}$$

A spherical solid is now seen to be the *best possible* shape for a GW detector. This is dictated by the optimality of the *overlap coefficients* $f^{(l,m)}_N$ between the universal form factors $\mathbf{f}^{(l,m)}(\mathbf{x})$ and the sphere's eigenmodes $\mathbf{u}_N(\mathbf{x})$, which also happen to have a *multipole structure*. **Only** the $l=0$ and $l=2$ *spheroidal* modes couple to GWs, whence *all the GW energy* is deposited into them, and only them.

A sphere is an ideal GW detector

GW coupled modes of a solid sphere



Sensitivity: Cross sections

Interaction ***cross section*** is defined by

$$\sigma_{\rm abs}(\omega) = \frac{E(\omega)}{\Phi(\omega)}$$

It can be shown that

$$\sigma_{\rm abs}(\omega_{nl}) = K_l(\aleph)\,\frac{GMv_t^2}{c^3}\,(k_{nl}a_{nl})^2 \qquad (l = 0 \text{ or } 2)$$

In particular,

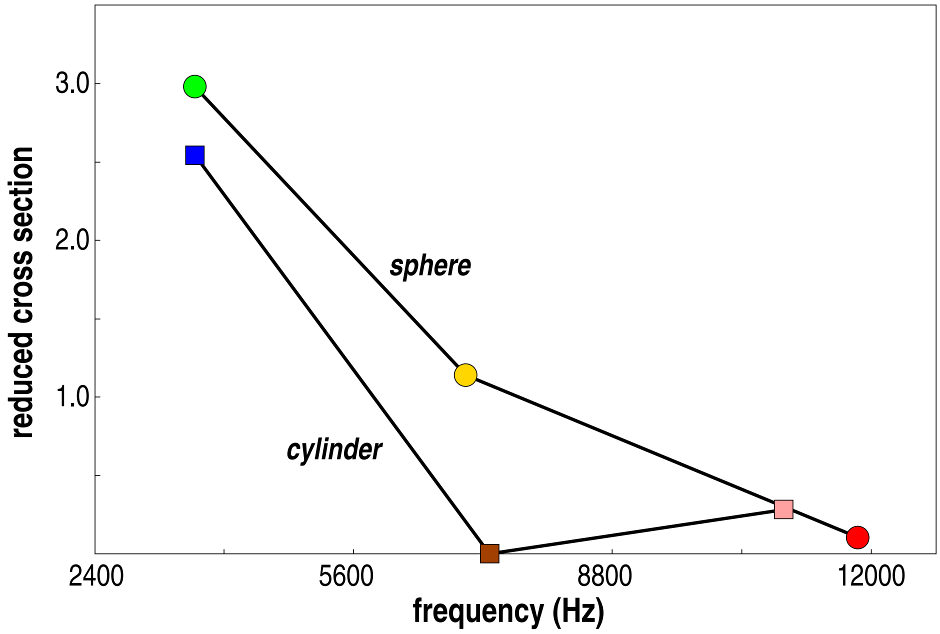
$$\aleph = {\rm GR} \Rightarrow \begin{cases} K_0(\aleph) = 0 \\ K_2(\aleph) = \frac{16\,\pi^2}{15} \end{cases}$$

and

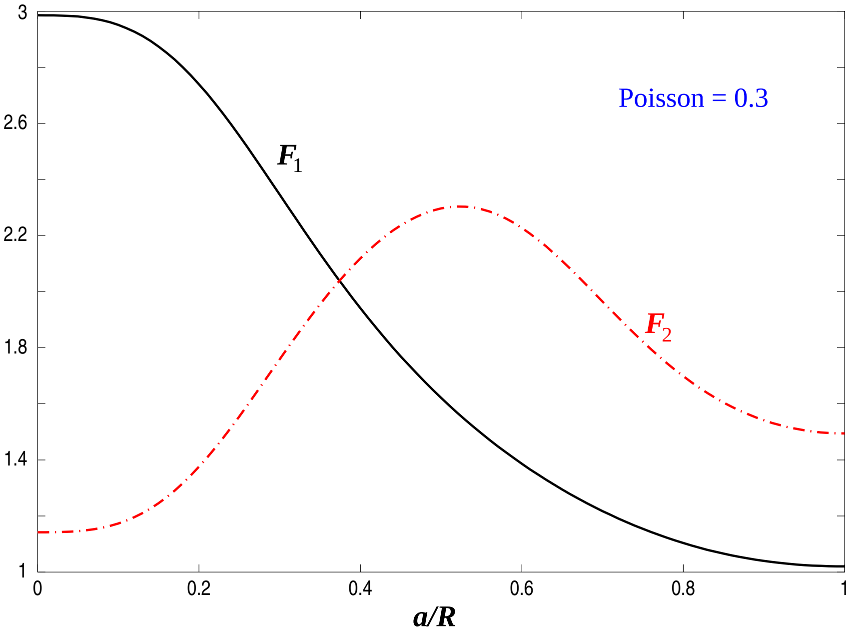
$$\aleph = {\rm BD} \Rightarrow \begin{cases} K_0(\aleph) = \frac{8\,\pi^2}{9}\,(3+2\omega_{\rm BD})^{-2}\,k\,\left[1+\frac{k\omega_{\rm BD}}{(3+2\omega_{\rm BD})^2}\right]^{-1} \\ K_2(\aleph) = \frac{16\,\pi^2}{15}\,\left[1+\frac{1}{6}\,(3+2\omega_{\rm BD})^{-2}\,k\right]\left[1+\frac{k\omega_{\rm BD}}{(3+2\omega_{\rm BD})^2}\right]^{-1} \end{cases}$$

where $\omega_{\rm BD}$ is the usual Brans-Dicke parameter, and k is a dimensionless number of order 1. As expected, in the limit $\omega_{\rm BD}\to\infty$, the GR result is recovered. A curious consequence of eq. (): the scalar field in Brans-Dicke theory is responsible for the excitation of *both* the (0,0) *and* the (2,0) antenna modes; the latter takes a up 1/6-th of the energy associated to such scalar field, while the remaining 5/6-ths go into the monopole modes.

In a ***solid*** sphere:



In a ***hollow*** sphere:

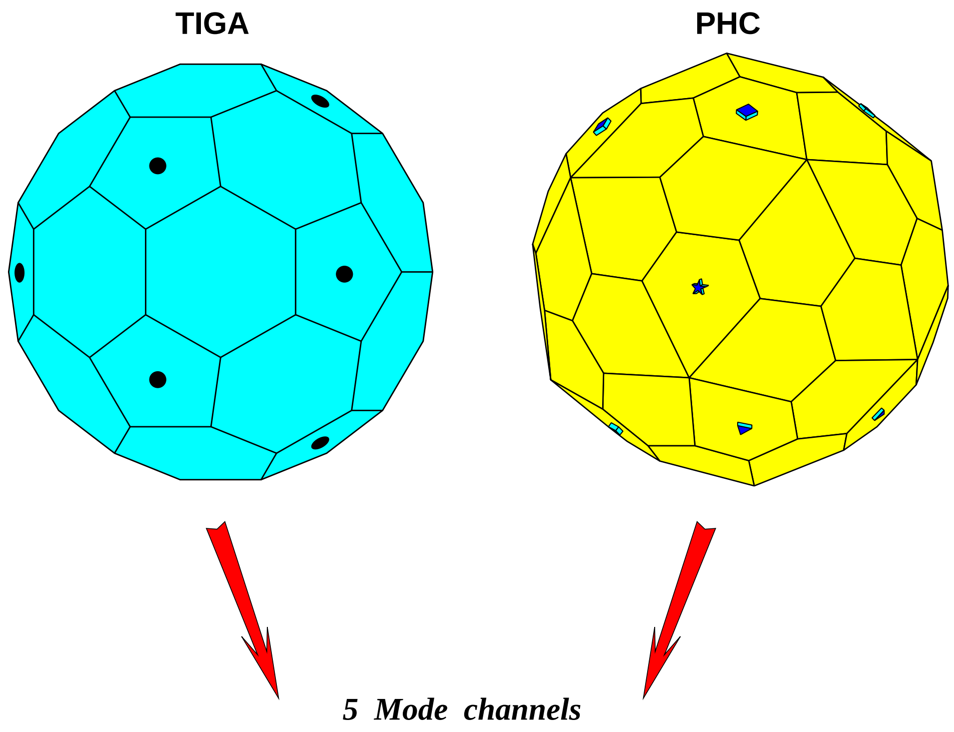


$$\sigma_{\rm Solid}(\omega_{12}) \sim 10^{-24}\,\pi R^2$$

Signal extraction in a spherical GW detector

Because a spherical detector has five ***degenerate*** quadrupole modes coupling to GWs (plus a non-degenerate monopole), its response must be ***multi-modal***, i.e., sphere's oscillations must be ***sampled*** at (at least) five different places (plus one, if monopole is to be also checked).

Conventional wisdom is based on *resonant transducers*, and special configurations are particularly useful:



where ***mode channels*** are readout combiations giving GW amplitudes:

$$\hat{y}^{(m)}(s) \equiv \sum_{a=1}^{5} v_a^{(m)*}\,\hat{q}_a(s) = \eta^{-1/2}\,a_{n2}\,\frac{1}{2}\left[\left(s^2+\omega_{m+}^2\right)^{-1} - \left(s^2+\omega_{m-}^2\right)^{-1}\right]\,\hat{g}^{(2m)}(s)$$

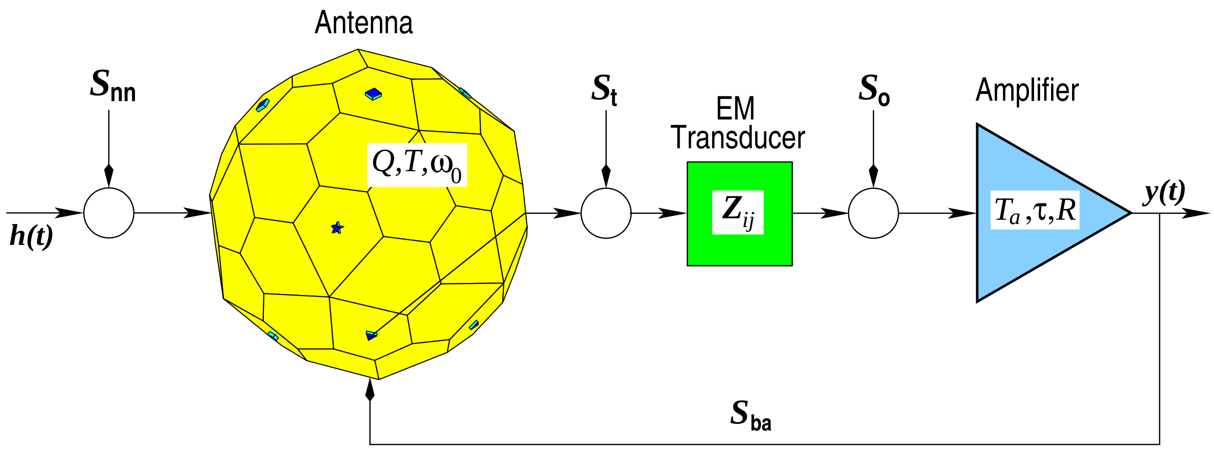
This is very elegant, but there are problems…

The motion sensing problem

OK, but *how do we* ***sense*** *the sphere's motions*?

A major problem is of course ***noise***: GW signal must *compete* with local & system disturbances —and ***must win the match***, if the detector is going to be of any use…

The general scheme is as follows:



where

- S_{nn} = Thermal noise in the antenna
- S_t = Noise in the transducer
- S_0 = Noise in the amplifier

A very important fact is this: there is ***no ideal readout system*** —if there was, an acoustic detector would have ***infinite bandwidth…***

The experimentalist's fundamental problem is thus to

reduce the noise in the whole detector system

Noise reduction problems

One should reduce the noise basically in the three stages the signal $h(t)$ traverses before becoming the human-readable output $y(t)$.

- Reducing the **antenna's** noise is conceptually simple: ***cool*** it down and try to find ***high Q*** materials. Cryogenics enables at present 10 mK for large masses; Q's can be as high as $10^6 - 10^8$.
- Reducing the **amplifier** noise is less simple: it has a ***fundamental limit***, given by the unavoidability of *quantum fluctuations*, which result in a temperature lower bound:

$$T_{a,\mathrm{QL}} \sim \frac{\hbar\omega_{\mathrm{GW}}}{k_{\mathrm{B}}}$$

  meaning that no energy in a GW of frequency ω smaller than a quantum $\hbar\omega$ can possibly be resolved.
- Reducing noise in the **transducer** is actually the most difficult problem:
  - It must be ***quiet***
  - It must ***efficiently convert*** mechanical into electrical signals
  - It must be ***impedance matched*** to the amplifier

Assuming everything goes as best it can, ***maximum sensitivity*** is

$$S_{h,\mathrm{SQL}}(\omega_{\mathrm{GW}}) = \frac{G\hbar}{c^3}\frac{4\pi\beta}{\sigma_{\mathrm{abs}}(\omega_{\mathrm{GW}})}$$

where

$$\beta = \frac{\Delta\omega}{\omega_{\mathrm{GW}}}$$

Non-resonant *vs.* resonant transducers

Perhaps the most immediate solution to the motion sensing problem is to attach a *small sensor* to the oscillating sphere —e.g. a ***piezoelectric*** ceramic— and let it *directly* convert its oscillations into electrical signals. This was actually the historical alternative developped by **J. Weber** in his first GW detection experiments in the 1960's.

Such transducer is usually:

1. A ***test mass***, i.e., it does ***not*** alter the large sphere's motions caused by GWs (or, indeed, any other agents).
2. A ***wide-band*** device, i.e., it responds to signals in a wide range of frequencies.

The first condition requires that

$$m_{\rm sensor} \ll M_{\rm sphere}$$

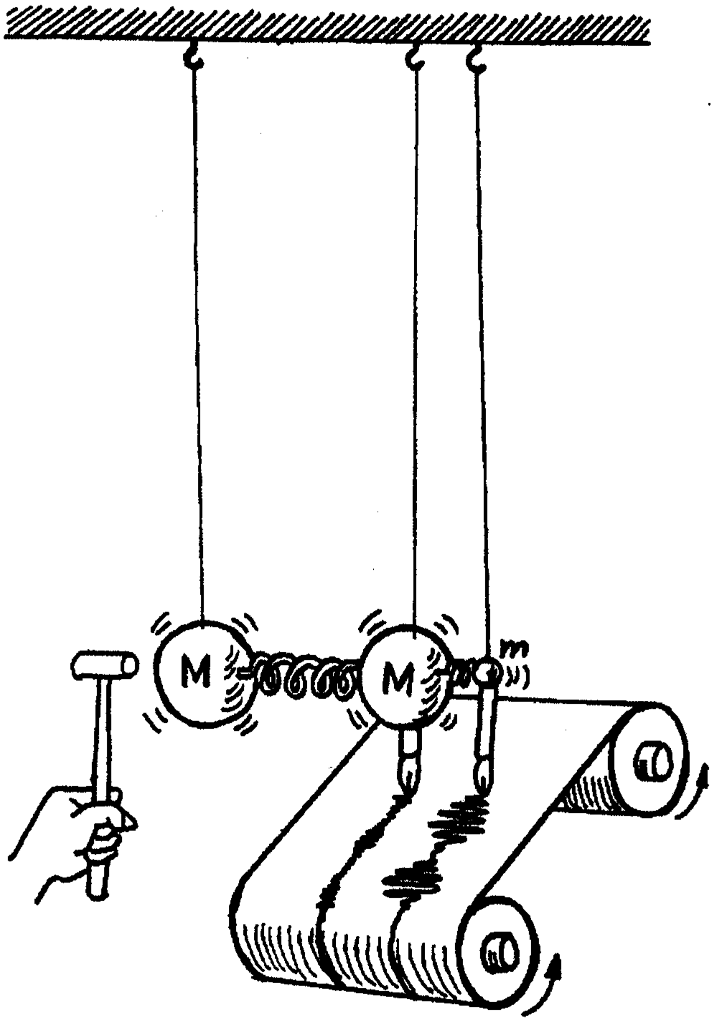
The second implies that such a sensor is rather ***noisy***, for the **spectral density** of *thermal* noise in a wideband system is given by

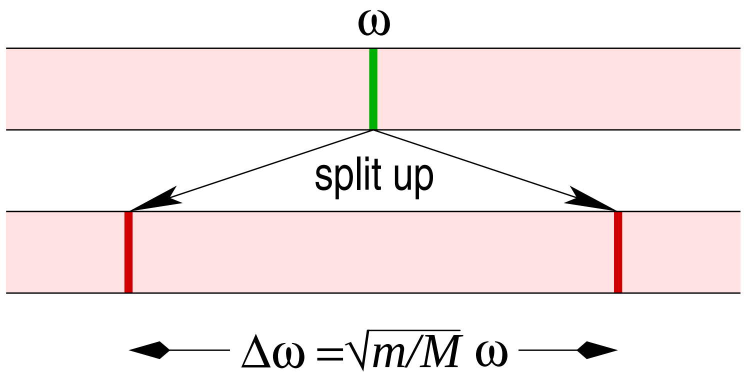
$$S_{\rm thermal} = \frac{2\zeta\, k_{\rm B}T}{m}$$

and therefore, being *inversely proportional to mass*, is much higher in the sensor than that in the sphere itself…

An alternative solution is the **resonant transducer**.

Resonant transducers are based on an elementary fact of classical mechanics:





Thus resonant sensors do actually ***couple*** to the large mass whose motion they intend to sense —even if they are *comparatively small*.

In fact ***joint dynamics*** matters for the sensing technique, and system motion is a ***beat***, both for the large mass and for the sensor, which is out of phase by 90° of the ***modulating*** sine wave.

In other words, ***resonant energy transfer*** happens back and forth between the large mass and the resonant sensor. This results in ***amplified*** oscillations of the smaller resonator:

$$\frac{A_{\rm sensor}}{A_{\rm sphere}} = \left(\frac{M_{\rm sphere}}{m_{\rm sensor}}\right)^{1/2}$$

This has one ***advantage***:

Enhances GW signal to exactly compensate for thermal noise

This is because ***all the GW energy*** is eventually in the sensor, not *part of it*.

But it also has a ***setback***:

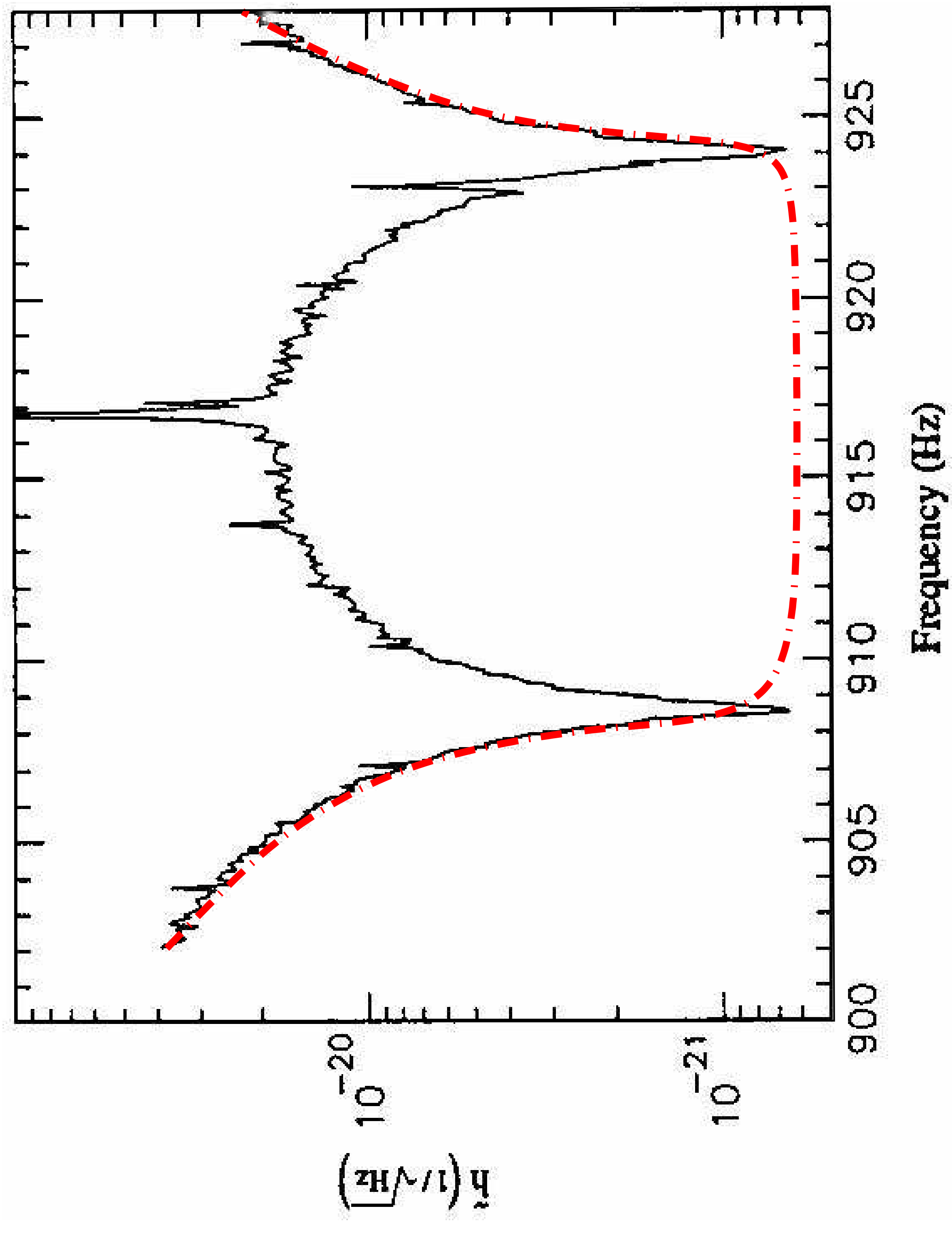
Bandwidth is fundamentally limited to $\Delta\omega$

This is because the GW energy is ***not all the time*** in the sensor, but only during shorter, periodic intervals of duration

$$\Delta\tau \sim \frac{2\pi}{\Delta\omega}$$

One can think of making the sensor ***larger***, but this destroys the scheme and, worse, poses serious problems of ***impedance matching*** with the amplifier.

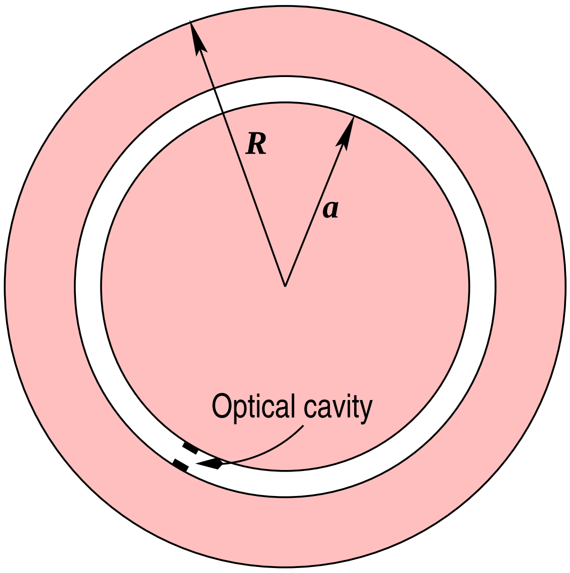
$NAUTILUS$ sensitivity



A novel solution: Dual Sphere

How to get rid of the ***band problem*** without falling into the noise problem of sensor noise?

The idea is to have a ***large non-resonant sensor*** which also captures the GW signal. For example:

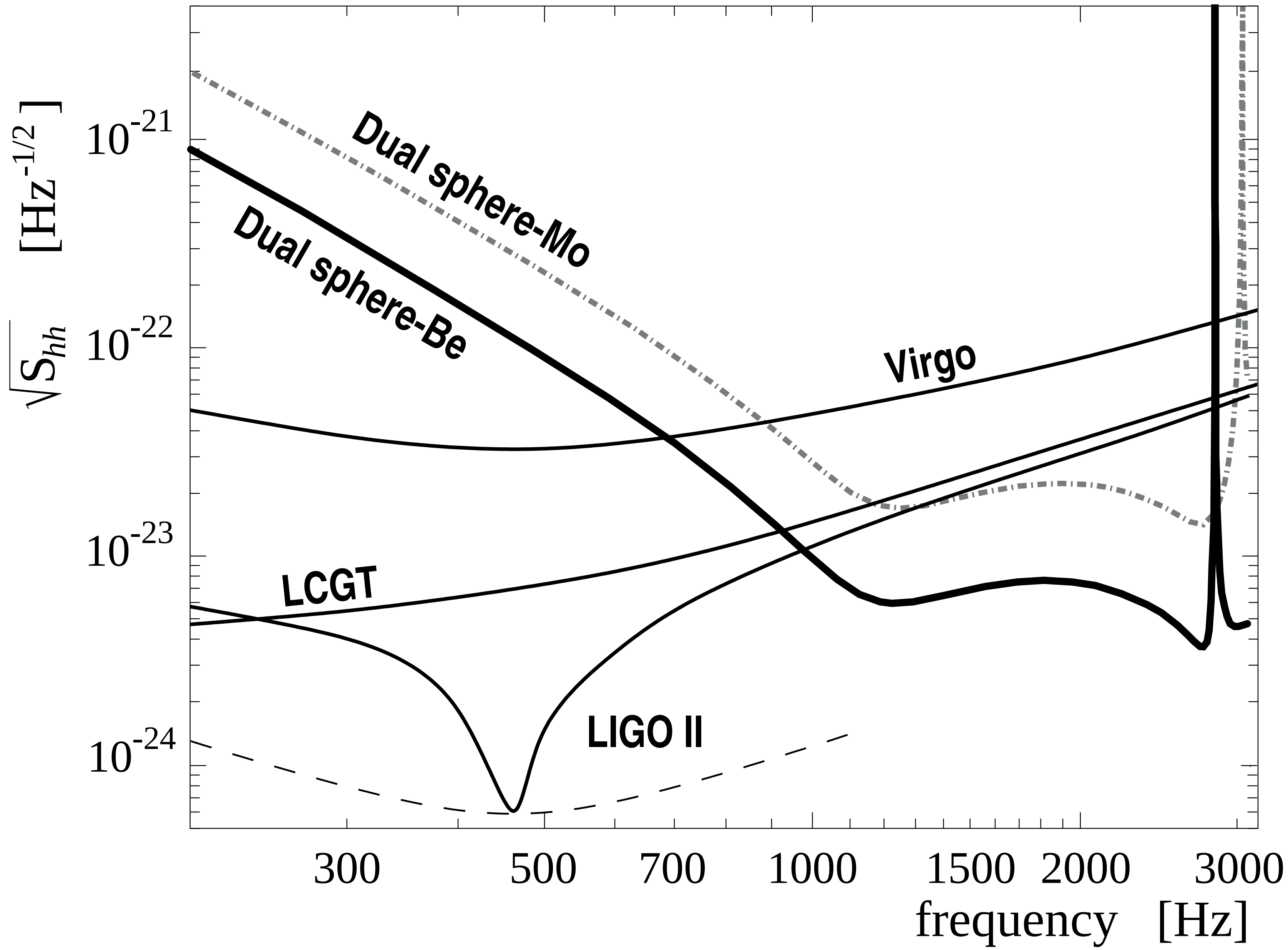


This one has a spectral density

$$S_{hh}(\omega) = \frac{S^{[th+ba]}_{uu,\mathrm{hollow}}(\omega) + S^{[th+ba]}_{uu,\mathrm{solid}}(\omega) + S^{shot}_{uu}(\omega)}{|u_{\mathrm{hollow}}(\omega) - u_{\mathrm{solid}}(\omega)|^2/|\tilde{h}(\omega)|^2}$$

which is reduced *between resonances* of the hollow and the solid spheres,

no signal cancellation!

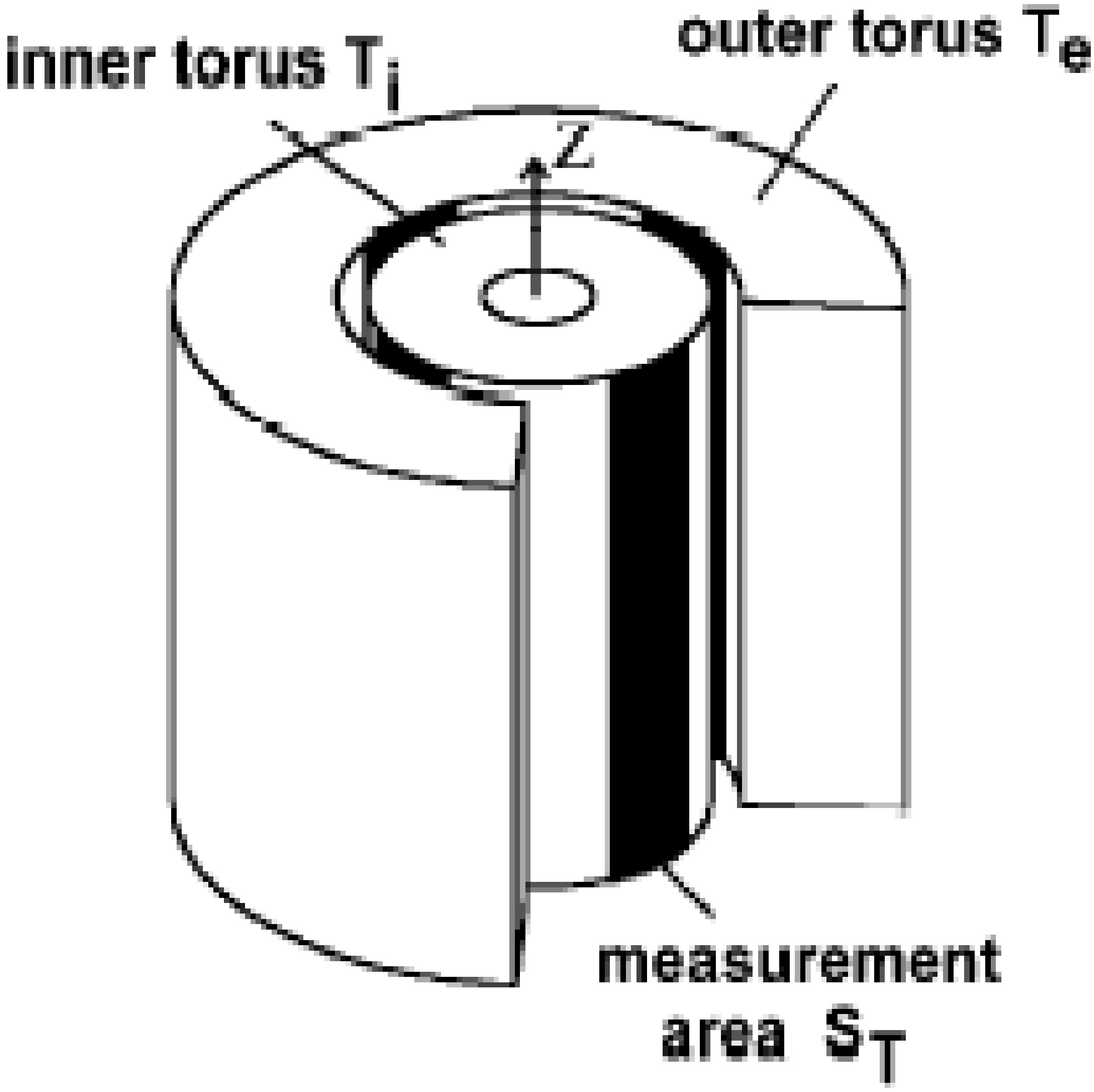


Dual sphere-Mo
Dual sphere-Be
Virgo
LCGT
LIGO II
10^{-21}
10^{-22}
10^{-23}
10^{-24}
$\sqrt{S_{hh}}$ [Hz$^{-1/2}$]
300
500
700
1000
1500
2000
3000
frequency [Hz]

Summary and conclusions

Spherical GW detectors are *conceptually* ideal, therefore a good promise for future development:

- They are *omnidirectional*
- Their oscillation eigenmodes uniquely *match* the structure of GW excitations, so they are most *efficient*
- Cross section is good at *second* quadrupole harmonic
- A single detector can measure *all six* GW amplitudes
- Different GW *theories* can be observationally put to test with such detectors
- A *single* sphere is capable of resolving the source *position* in the sky
- *Bandwidth* can be enlarged with a new concept *nested sphere*
- *Technology* for their construction is based on sound experience from cryogenic bars
- Progress in underway (Italy-France-Spain) to solve building problems — ***dual toroidal*** antenna


inner torus T_i
outer torus T_e
z
measurement
area S_T