Rewriting the exact solutions book (and how we handled the cosmological solutions)

Malcolm A.H. MacCallum
School of Mathematical Sciences,
Queen Mary, University of London,
Mile End Road, LONDON E1 4NS, U.K.

Abstract

The history of the exact solutions book is recounted and the decisions made in its evolution discussed. In particular the decisions made for the second edition, its contents and layout, are described. As an example, fuller details are given of one of the chapters involving solutions interpretable as cosmologies.

1 Introduction: the genesis of the book

Talking about the writing process seems a little self-centred, to put it mildly, but it took up most of my research time for a considerable period so I feel a seminar on it is justifiable. In the following I try to preserve the flavour of the talk and in particular I will not give full references which can be found in the book itself [1]: they are listed here by authors and date.

My involvement with the topic started with some hand-written lists of references collected by George Ellis, which I saw when I was one of his research students, and which gave classified lists of exact solutions. I started to collect references myself, on file cards, with the thought that maybe one day, when I could not think of anything better to do, I might write a review or book about them. This project seemed suitable for the end phase of a career.

So I was very pleased when I received a letter from Jena asking for information and offprints concerning exact solutions. Dietrich Kramer and Hans

Stephani had published a review in 1972 in German (Kramer, Neugebauer and Stephani 1972), and, I understand, realised it had not been widely read. They set out to do a fuller version in English, and had written to a wide circle of workers in the field for information. I happily offered a copy of my file cards and thought "Hooray, somebody else is taking on this thankless task. Good luck to them".

In 1976 Stephani visited London as part of an ongoing series of exchanges between Jena and Bill Bonnor's group, and we met, I think for the first time[1]. It emerged that the Jena authors wanted a native English speaker to edit the text, and after some thought I accepted their invitation, later, of course, becoming a full co-author (I was the primary author of chapters 8-12 of the original edition and I also assisted the negotiation of the Western edition by representing the Cambridge University Press at a meeting with the German publishers in Berlin).

The bibliography for the first edition was kept on file cards with holes punched around the edges according to a classification scheme which allowed cards relevant to a particular topic to be extracted using the holes. The reprints themselves had been diligently collected and filed with the help of graduate students paid for by the Jena department. The text was handwritten and given to secretarial staff to type. It was then exchanged between Jena and London, a process that took about 6 weeks initially, presumably due to scrutiny by security on both sides of the Iron Curtain, but speeded up to a week or so by the end. It has been widely cited, though not always understood. (Some of the time of the book's authors has been spent as referees correcting the misapprehensions of authors of papers claiming to have found new solutions in classes where all solutions are already known, and so on.)

In the first edition a number of major decisions came from discussions between the authors. The two main groups were

1. The overall layout. This was in 5 parts covering

   Part 1 Mathematical background and classifications of solutions

   Part 2 Solutions classified by groups of motion

   Part 3 Algebraically-special solutions

[1] Hans, who sadly died on September 14th, 2003, became one of my closest colleagues and a good friend.

Part 4 Solutions obtained by other means (almost all solutions belonged under Parts 2 or 3)

Part 5 Tables

2. Restrictions on the energy-momenta considered. Those allowed were

   - Vacuum
   - Electrovac
   - Perfect fluid
   - "Pure radiation" (also called null dust or sometimes null fluid).

   In particular we excluded

   - Combinations of the above cases
   - Spinor fields (Dirac, Yang-Mills etc)
   - Scalar fields (except, implicitly, massless fields whose energy momenta were the same as "stiff" fluid)
   - Charged fluids.

The latter restrictions could be regarded as laziness or inertia, but were a good decision in restricting us to classes which could be treated in a systematic way and to a manageable total amount of work – and it leaves something for others to do!

2 A new edition?

On the fall of the "Iron Curtain", the original publisher, Deutscher Verlag der Wissenschaften, was sold to Hüthig who were not interested in publishing a new edition. As a protective measure Hans Stephani wrote to the publishers asking about this, because the contracts we had for the first edition said the rights would revert to us if no positive response was received within six months, which duly happened. We then found that the other original publishers, Cambridge University Press, were keen to do a new version and that their warehouse was running out of copies of the book (it went out of print in the late 1990s).

So we met some time in 1991 or 1992 (I'm not sure exactly when) and discussed the idea and agreed to do it. We then began collecting references. Having found that about 95% of the references quoted in the first edition had come from use of Physics Abstracts to locate bibliography, we restricted ourselves to that source, using the FIZ Karlsruhe database, and, later, the ArXiv preprint archive, as well as anything that came to our attention serendipitously.

The first phase of the work was the bibliographical data collection, the collection of copies or reprints, for which we again used graduate student labour, and the formatting of the old references in the electronic format of the other data. There were many references. In the end we collected about 4000 new ones (since 1980), to add to the 3000 or so we had studied for the first edition. The cutoff for systematic collection was 1999 but some later papers are cited: we leave systematic treatment of the new millenium to our successors. Of those 4000, Hans Stephani read about 2000 with help from Eddie Herlt, I read about 900, Dietrich Kramer had about 600 to read and the rest went to Cornelius Hoenselaers. Some were read by more than one of us. We had to be brutal in rejecting papers which did not give first discoveries or which failed to give relevant or complete solutions, and in omitting methods or commentary of relatively low utility in our view (which the original authors will not share!).

Some early decisions for the new edition were

- the basic structure was retained
- LaTeX and STN were to be the basic tools
- we would retain the same limitations on energy-momentum, in particular

  no massive scalar fields
  no charged fluids
  no mixtures
  no hairy black holes

- the generating techniques chapter would be rewritten completely (in fact, some old material was reused and the chapter split in two)

The following new chapters were needed

[1] Lie symmetries of differential equations
[2] Homothetic motions
[3] Inhomogeneous cosmologies
[4] Colliding waves
[5] Special subspaces (including intrinsic symmetry)

[6] Equivalence problem
[7] Other generating techniques
[8] And a catch-all chapter

The decision that the new version would be produced in LaTeX (some of us used Scientific Word to write it) implied the production of a LaTeX version of the first edition. (Unfortunately, the idea of a Web version, which might have been a serious alternative were we starting now, did not occur to us then.) Andreas Koutras did a great deal of the work transforming the old edition into electronic form and played a crucial role at this phase (during 1995-6): he also checked many of the solutions in the original book. To complement this we had to format the references in BibTex format, a task which I took care of.

In 1996 I spent 3 months in Jena as a Gästprofessor. During this time we set up scripts to automate transfer from the FIZ format STN to BibTeX and devised conventions for our LaTeX input in particular a cross-referencing scheme which enabled us to label material without conflicts arising (one part of this being that all the elements in the first edition retained their original numbers as labels), and started to use the publisher's macros. It was really only after this that the bulk of the old material was recast into electronic form. At this point the new work had largely consisted of rough classification of new bibliographic material by topic (Stephani did most of this) and some preliminary reading. From then on there was a great deal of hard work in writing: for me personally it took most of my research time in 2000-2 as well as significant time before that.

After the rough classification allocating each reference to certain chapters, we worked by reading and if necessary re-classifying the papers (for which process we had to devise database maintenance and sharing procedures, largely administered by Hans Stephani with help from me). The draft chapters were kept, dated and signed, in a central repository to which we could each upload as we updated.

Other relevant books appeared before or during our writing and these somewhat affected our plans in that we were able to refer readers to them for additional information, especially on interpretation and the physical significance of the known solutions in some larger picture. The most important of these were Griffiths' 1991 book on colliding gravitational waves, Krasinski's 1997 book on inhomogeneous cosmologies, a veritable tour-de-force which went wider than our book in considering exotic energy-momentum tensors but was more restrictive in that the solutions considered had to have a Robertson-Walker limit, the book on dynamical systems in cosmology edited by Ellis and Wainwright (1997), which focused mainly on the spatially homogeneous and related models, Bicak's contribution to the Ehlers Festschrift (2000), which was selective among solutions but went much further in interpretation especially of radiative cases, and the book on gravitational solitons by Belinskii and Verdaguer (2001).

An author was added, Cornelius Hoenselaers joining us to relieve the pressure on Dietrich Kramer (and eventually the order of the authors was altered to reflect the overall input of each to the new edition, including work copied from the first edition).

3 What became of the old chapters

Every chapter has been thoroughly revised: in all the chapters describing the actual solutions, many new solutions were added, and some old but overlooked solutions and references too.

A few chapters were shortened though the whole book is much longer (just over 700 pages compared with 400 odd: the cost per page at full price is about twice the price of the 1980 edition, which is not bad for inflation over that period!).

Not all solutions are now given explicitly, because the number of solutions made this impossible. Some are now simply given as references.

In Part 1, the background section, chapters 1-8 became – chapters 1-8! However, some material was reordered, and material was added on fibre bundles, junction conditions, integrability of tetrad methods, the GHP formalism, homotheties and so on. Two new chapters were added as described below.

In Part 2, on classification by groups, chapters 9-20 became chapters 11-22. Material was added on homotheties, solutions found by dynamical systems methods, multipole moments, the factor structure in stationary axisymmetric solutions, and the boost-rotation case, and the order of the material on spherically symmetric solutions was changed with some reordering between chapters. Again there were two new chapters (see below), and chapter 21 became chapter 24. One problem arising in this Part was where to allocate all the new work on solutions with a G_2 acting on S_2 which could be regarded as cylindrical, cosmological or colliding wave solutions, and which also gave examples of applications of generating techniques.

In Part 3, on algebraically special solutions, the chapters were renumbered to run from 26-33 rather than 22-29 and many new solutions were added.

Part 4, which dealt with other methods, was more extensively reorganized than other parts. Chapter 30 on generating techniques, as remarked above, was almost totally rewritten as chapter 34. In chapter 31 (now 35) on special vectors and tensors, collineations were added. Chapter 36 was introduced and chapter 32 became chapter 37.

Finally chapter 33, consisting of tables, was extended.

The one old/new chapter, now chapter 34, is on applications of generating techniques and is organized with the following main parts

- solutions found by harmonic maps
- solutions found by the HKX method, arising from the prolongation for the Ernst equation
- Bäcklund transformations
- the Belinskii-Zakharov method.

Of these, the first and last gave especially large tables of references to actual applications. The remaining methods are only briefly described as there was not enough room to expound all them in full. This is also the place where the bulk of the solutions found (which in principle is a list as long as one wishes to make it by applying the known methods) has to be given only in the sense of references.

4 What became of the new chapters?

The new chapters mentioned above were given working numbers 40-46 and 49. I shall describe what happened to them in the order of their working numbers.

4.1 Chapter 40

This chapter concerned the equivalence problem, i.e. the problem of deciding whether two solutions are locally isometric. To describe the method in full would be a separate lecture, so I will just note the outline of the resulting chapter. It includes an (almost) up-to-date review of work on scalar polynomials, limits of differentiation required, and other research, old and newer, of mine and of or with Åman, Joly, Paiva, Skea and others. Other features are:

- it includes sections from the first edition on scalar polynomial invariants and Petrov classification
- the central section is on the Cartan method and the theory of classification algorithms
- all references to computer algebra programs were omitted except acknowledgement to those used, CLASSI and REDUCE, and some references to reviews
- it includes a section on limits of spacetimes.

This became chapter 9.

4.2 Chapter 41

This started as a chapter on Lie point symmetries. Extension to generalized symmetries led on to the Kramer-Neugebauer and Geroch approches to generating methods, but is also related to some methods for perfect fluid solutions, Kerr-Schild, and other cases, in principle. As the strength of the connection with the theory behind the generating techniques for solutions admitting a G_1 or G_2 became more apparent, we decided to merge the draft with the relevant part of the former chapter 30, as rewritten. The new draft then included a discussion of prolongation (and isovectors, conservation laws and pseudopotentials), which may now be the most mathematically demanding part of the book, linearized methods (Kinnersley-Chitre and the Geroch group), and in brief form, Bäcklund transformations, Riemann-Hilbert methods, harmonic maps, variational Bäcklund transformations and Hirota's method. We added other generating methods (as in an expanded version of the old section 30.6), in particular conformal transformations, and the methods for perfect fluids due to Racz and Zsigrai, and Krori and Nandy, the method of Wainwright, Ince and Marshman which gives stiff fluid solutions from vacua (this had been in chapter 15, with related ideas of Belinskii and Letelier), and the methods of Ehlers and Buchdahl. This became chapter 10, the last in Part I.

4.3 Chapter 42

This new chapter on special subspaces (mostly hypersurfaces) contains a general introduction, and sections on solutions with flat slices (as discussed by Stephani, Wolf and others), sections with conformally flat slices, and other intrinsic symmetries. It became chapter 36.

4.4 Chapter 43

This was the catch-all chapter, but in the end all papers assigned to this chapter because they did not apparently relate to any existing topic were reassigned elsewhere.

4.5 Chapter 44

This was to be a chapter on homotheties. It covered homothety groups – the general theorems and their enumeration, the possible metric forms with homothety (not yet solutions), and a survey of solutions. These parts were in the end added to the former chapters 8 and 9 (now 8 and 11).

4.6 Chapter 45

This chapter was on inhomogeneous cosmologies. It was one of the chapters I was responsible for and I describe below at greater length how it developed as an example of the writing process. It became chapter 23.

4.7 Chapter 46

This chapter on colliding waves involved a major writing effort since there had been many papers and a book (Griffiths, 1991). It includes

- an expanded version of the introduction from the old chapter 15
- the relevant forms of the vacuum field equations
- vacuum cases with collinear polarization
- vacuum cases with non-collinear polarization
- Einstein-Maxwell cases
- stiff fluids and pure radiation

In the last section mentioned it was necessary to revisit a 1980s controversy between Chandrasekhar and Xanthopoulos, Feinstein, Senovilla and myself, and a number of other authors, about the significance of the fact that the gravitational field equations do not uniquely specifiy the energy-momentum in the interaction region. Our wording here was chosen rather carefully! It became chapter 25.

4.8 Chapter 49

This chapter was to be on generating methods other than those for solutions with a G_1 or G_2 and included the old section 30.6. It was incorporated in other chapters.

5 The writing of the chapter on inhomogeneous cosmologies

This was the last chapter to be written. The first cut was that it should be ordered by group types in order, i.e. solutions with H_3, G_2 (only), H_2, G_1 and H_1, where H_n denotes a homothety group and necessarily contains a G_{n-1}. After some discussion, it was decided the chapter should only include perfect fluid solutions and that the related vacua should appear elsewhere. Then, recognizing that not all such fluid solutions can reasonably be considered cosmological, the chapter was retitled "Inhomogeneous perfect fluid solutions with symmetry".

It was decided the chapter should include the general forms of the solutions obtained by the Wainwright, Ince and Marshman method which generates solutions with stiff fluid; in the old edition only the specializations with a G_3 had been given in full. Now the detailed H_3 or G_2 on S_3 solutions are given.

Many papers giving stiff fluid solutions obtained from vacua or obtainable in that way had appeared. The applications of the Wainwright, Ince and Marshman method itself were listed in what became chapter 10, while others which could have been so obtained were given, but only as references, in chapter 23.

Quite a few solutions found are definitely unphysical e.g. they violate energy conditions. These were ignored. Many more, almost all, are non-barotropic fluids (i.e. one cannot write $p = p(\mu)$): these were kept because we know so few inhomogeneous fluid models that even these may be useful in cosmology.

There were classifications of possible solutions. Wainwright (1979) gave a very general scheme but almost all the cases described were unpopulated by actual solutions. In 1981 he gave a scheme for the more restricted class with a commuting G_2 and orthogonal flow: this has been widely used, its classes are populated (indeed it covers all explicitly solutions known to belong in this chapter except Kramer's) and it was used in the final writeup. A similar scheme for non-Abelian G_2 was devised by Sintes but few such solutions are known.

Some papers considered contained, or appeared to contain, spatially homogeneous solutions. These were read first because they were needed for chapters 11 and 12 (new numbers).

In total there were about 165 papers to read for this chapter. About 20 of these were eliminated early on, mainly because of misclassification or because the energy-momentum tensor did not belong or there was no solution, only equations.

I knew that some important work had been done by Senovilla and his students, by Mars in his thesis (of which I was an examiner) and later, in Vera's thesis, by Carot and his colleagues and in Sintes' thesis, and earlier by Koutras in finding solutions with a homothety in the first edition whose symmetry had not been noted. I started with these references. The solutions were checked, using CLASSI and REDUCE. This led to a subdivision into those solutions with diagonal Abelian G_2 form, non-diagonal solutions with orthogonal transitivity, and solutions without orthogonal transitivity. No section for the non-Abelian case of G_2 was needed because all explicitly known solutions with such a group have a G_3 on V_2.

Reading the references just picked out suggested that the main methods for finding solutions had been: assuming separability of the metric coefficients, whether in comoving or non-comoving frames; assuming additional conformal symmetries; and, for the H_3 cases, dynamical systems methods.

Putting these ideas together led to an outline as follows.

23.1 H_3 on S_3

Including the Wainwright, Ince and Marshman solutions

23.2 H_3 on T_3

Including solutions due to Uggla

23.3 G_2I on S_2

23.3.1 Diagonal cases

Separable in comoving coordinates (Senovilla and Ruiz)
Partially separable (Mars, Roy and Prasad)
Separable in non-comoving coordinates (Vera)
Conformal motions (Carot et al, Mars and Wolf)

23.3.2 Non-diagonal cases, orthogonally-transitive

Including work of Mars and Senovilla

23.3.3 Non-orthogonally-transitive cases

Including work of Wils and of van den Bergh. All of these are stiff fluid solutions but are not obtainable by the Wainwright et al method: instead they are found by separability assumptions.

23.4 Solutions with a maximal H_2 or smaller groups

In the final event, this subsection was handed over to Hans Stephani but there proved to be very little to say beyond references to special cases treated elsewhere.

The next steps were to put the Senovilla and Ruiz classes into as simple a form as possible (which was not trivial) and similarly the Roy and Prasad solutions. Then I tackled the Vera, and Senovilla and Vera, solutions: these have a subclassification based on the numbers of independent functions of each of the two remaining variables. Only a few of the possible cases have been completely integrated. Next came a similar study of the Mars and Senovilla cases. Throughout this phase it was important to keep a lookout for duplication of known solutions (Allnutt's type D solution is one which recurs) and for additional homothetic motions.

Some of the main problems are outlined in the following paragraphs.

Uggla's 1992 solutions with an H_3 all have a G_2. Where do they fit among the G_2 solutions? This was not easy to work out due to the very complicated form in which the solutions naturally arose from Uggla's dynamical systems approach – they have numerous interrelated constant parameters. Eventually they turned out to be special cases of the Senovilla-Ruiz class (which had been given in a simpler form) together with one special subclass overlooked by Senovilla and Ruiz. For example, the solution given in the first column of table 1 of Uggla (1992) turns out to be a special case of the Senovilla and Ruiz class labelled as $2m = 1 + n$. (In the talk I showed the details of these solutions.)

The second set that were hard to check were the cases with conformal motion. This is because the metric forms obtained had been derived from the allowable coordinate forms for the conformal symmetry groups under different assumptions hard to reconcile. In particular, how did the assumptions of the two main papers intersect? It turned out that two classes of solutions were duplicated and/or generalized one another and/or differed by the interchange of sinh and cosh or similar trivial changes. One class in the Carot et al paper is also in that of Vera. An example given in the talk is that the solution of equation (140) of Carot, Coley and Sintes (1996) is conformal to a 2+2 decomposable solution (like the "Bertotti-Robinson" metric, more correctly ascribed to Levi-Civita): this implies it admits a C_6 rather than the C_3 assumed.

The final example of problems was the many other papers with special cases (mostly older papers): here the effort was to identify them and fit them into place in the main classes. Examples are given in the chapter but a number more papers are not cited because they amount to rediscovery of (usually, special cases of) already known metrics.

6 Conclusion

I hope that by describing the processes we followed to plan and write the new edition of the book, I have given some insight into the process of producing such a work, and into the things people entering this field should consider before claiming to have found a new solution.

References

[1] H. Stephani, D. Kramer. M.A.H. MacCallum, C.A. Hoenselaers and E. Herlt, *Exact solutions of Einstein's field equations* (Cambridge University Press, Cambridge, 2003).