Quasilocal mass

in General Relativity

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$\boxed{\text{Plan of the talk}}$

I. Why we need quasilocal mass/energy (qlm)?

II. Some ways of defining quasilocal mass (Bartnik, Bray, Brown – York, Epp, Liu – Yau).

III. Some properties of proposed quasilocal masses and/or associated foliations.

Liu and Yau on qlm:

It (qlm) can be used to define binding energy of stars. It can also be useful for numerical relativity as it tells us how to cut off the data on a noncompact to a compact region where the quasilocal mass of the compact region approximates the ADM mass of the noncompact region.

When we evolve the data according to Einstein equations, we need local control of energy to see how the space changes. The positivity of quasilocal mass is essential for such an investigation. This could be regarded as a generalization of the energy method in the theory of a nonlinear hyperbolic system

Liu and Yau, Phys. Rev. Lett. (2003)

Quasilocal mass/energy definitions: preliminaries.

Needed elements: space-time Ω, Cauchy hypersurface Σ, 2-surface $\delta\Sigma$ (that bounds a segment of Σ);

extrinsic curvatures: K_{ab} - of the embedding $\Sigma \in \Omega$; p_{ab} of the embedding $\delta\Sigma \in \Sigma$.

$$q = tr_{\delta\Sigma} K_{ab};$$

$$p = tr_{\delta\Sigma} p_{ab}$$

Alternative description: optical scalars

$$\theta_+ = p + q$$

$$\theta_- = p - q.$$

Brown, (Lau) and York quasilocal energy definition (PRD (1993))

Let $\delta\Sigma$ be given a positive internal curvature (twice the Gauss curvature);

embedd $\delta\Sigma$ in a Euclidean subspace of the Minkowski spacetime. (This is unique up to an isometric transformation and yields a trace of the extrinsic curvature p_0 of the embedding.) Then:

$$\boxed{E(\Sigma, \delta\Sigma) = \frac{1}{8\pi} \int_{\delta\Sigma} (p_0 - p)}$$

Comments

The internal curvature positivity condition is essential;

$$E(\Sigma, \delta\Sigma) \to m,$$

(if K_{ab}) falls off quickly enough); m - the asymptotic mass.

B(L)Y mass in spherical spacetimes

in maximal foliations of the Schwarzschild geometry

$$\boxed{E(\Sigma, \delta\Sigma) = R(1 - \sqrt{1 - \frac{2m}{R} + \frac{C^2}{R^4}});}$$

clearly the energy depends on a slice! (and on a slicing condition)

at the event horizon $2m \geq E(\Sigma, \delta\Sigma) = 2m(1 - \sqrt{\frac{C^2}{16m^4}}) \neq m$;

at spatial infinity $E = m$.

Spherical systems with matter: generally the energy is not monotonic inside regions with matter. That is not a problem in the case of spherical symmetry (since $2m > E_{B(L)Y} > \epsilon m$ and m is *essentially* the energy norm outside regions with horizons),

but severely diminishes the probability that $E_{B(L)Y}$ can be treated as an energy norm in nonspherical geometries.

(Epp), Liu and Yau quasilocal mass definition (PRD (2000); PRL(2003)

Let $\delta\Sigma$ be given a (positive) internal curvature (twice the Gauss curvature) and optical scalars θ_+, θ_-;

embedd $\delta\Sigma$ in (a Euclidean subspace of) the Minkowski spacetime. (This yields a trace of the extrinsic curvature p_0 of the embedding.) Then:

$$\boxed{E(\delta\Sigma) = \frac{1}{8\pi} \int_{\delta\Sigma} (p_0 - \sqrt{\theta_+ \theta_-})}$$

Comments

Liu and Yau demand the positivity of the internal geometry of $\delta\Sigma$. The embedding can be made into a Euclidean subspace of the Minkowski spacetime and it is unique up to an isometric transformation. Epp does not assume this positivity; the embedding is now not unique.

$$E(\delta\Sigma) \to m,$$

m - the asymptotic mass.

The qlm is now assigned to a 2-surface, no reference to a Cauchy hypersurface. (That is not very helpful.)

The qlm is invariant under Lorentzian boosts.

LY mass in spherical spacetimes

in maximal foliations of the Schwarzschild geometry

$$\boxed{E(\delta\Sigma) = R(1 - \sqrt{1 - \frac{2m}{R}};}$$

Given two-surface, the energy does not depend on a slice! (or a slicing condition)

at the event horizon $2m \geq E(\delta\Sigma) = 2m$;

at spatial infinity $E = m$.

Spherical systems with matter: generally the energy is not monotonic inside regions with matter. That is not a problem in the case of spherical symmetry (since $2m > m_{B(L)Y} \equiv E(\delta\Sigma) > m$ and m is *essentially* the energy norm outside regions with horizons),

but it would be difficult to show that $m_{B(L)Y}$ can be treated as an energy norm in nonspherical geometries.

Hawking quasilocal mass definition (JMP (1968))

Let $\delta\Sigma$ of an area S be given optical scalars θ_+, θ_- and an internal curvature $2K$;

Then:

$$\boxed{m_H(\delta\Sigma) = \sqrt{\frac{S}{64\pi}} \times \frac{1}{16\pi} \int_{\delta\Sigma} (4K - \sqrt{\theta_+ \theta_-})}$$

Comments

This qlm is now assigned to a 2-surface, no reference to a Cauchy hypersurface, but in fact the Hawking mass definition makes sense only in special foliations (of a Cauchy hypersurface in a spacetime). It would not be reasonable to assign m_H to a 2-surface chosen by random. (The only surface in the Euclidean space that would give $m_H = 0$ is a metric sphere.)

$$m_H(\delta\Sigma) \to m,$$

m - the asymptotic mass.

This qlm is invariant under Lorentzian boosts.

Hawking mass in spherical spacetimes

in any foliations of the Schwarzschild geometry

$$\boxed{m_H(\delta\Sigma) = m;}$$

i) The energy does not depend on a slice! (or a slicing condition)

Spherical systems with matter: the energy is **monotonic** inside regions with matter, under reasonable energy conditions (the dominant energy condition on a given Cauchy slice). Proof: inspect the volume representation of the Hawking mass

$$\boxed{m_H(\delta\Sigma) =}$$

$$\boxed{2\pi \int_{V(\delta\Sigma)} dV r(\rho \frac{\theta_+ + \theta_-}{2} + j \frac{\theta_+ - \theta_-}{2}).}$$

Hawking mass in general spacetimes

Define

$$M(V(S)) = \int_{V(S)} dV \frac{(\theta_+ + \theta_-)}{pN}\rho, \tag{1}$$

$$J(V(S)) = \int_{V(S)} dV \frac{(\theta_+ - \theta_-)}{pN} j_i n^i, \tag{2}$$

$S_A = K_{Ar}$ (A=1,2), $M_H(S_i)$ be the Hawking mass assigned to a surface S_i (i=1,2), and (the Geroch flow condition) $Npr/2 = 1$.

$$\begin{aligned} M_H(S_2) = M_H(S_1) + M(S) + J(S) + \frac{1}{16\pi}\int_{V(S)} dV [\\ \frac{2}{N}((\frac{\nabla^{(2)}_L N}{N})^2 + \frac{r}{2N}(\theta_+ - \theta_-)S^L \nabla^{(2)}_L N + S^A S_A) + \\ \frac{pr}{2}(K^{TA}_B K^{TB}_A + p^{TA}_B p^{TB}_A - 2\frac{(\theta_+ - \theta_-)}{2p} K^{TA}_B p^{TB}_A)] + \\ \frac{1}{16\pi}\int_{V(S)} dV 2\frac{(\theta_+ - \theta_-)}{2pNr}\nabla^{(2)}_L(\frac{S^L}{N}). \end{aligned} \tag{3}$$

Can one make from this volume integral an energy norm?

1. The integrand can be made strictly positive, provided that there exists a foliation such that θ_+/θ_- is constant on foliation leaves (EM, Mars, Simon, PRL(2002)) – a slicing condition. (A special case - moment of time symmetry initial data, $\theta_- = \theta_+$ – has been proven (Ilmanen and Huisken).)

2. If this slicing condition is satisfied, then the matter related terms constitute an energy norm outside horizons.

3. In order to identify here an energy norm for gravitational waves, one would have to identify the "true" degrees of freedom (in the spirit of Jezierski – Kijowski (1990's)).

Conclusions