On the monotonicity of the Hawking mass and the Penrose inequality for general horizons.

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Abstract

Huisken and Ilmanen's method for proving the Penrose inequality in the Riemannian case involves flows of surfaces for which the Geroch-Hawking mass is monotone. It is natural to explore similar ideas for the general spacetime setting, i.e. whether there are flows in spacetime for which the Hawking mass is monotone. We analyze this issue and find necessary conditions on the flow, and on the surface, to ensure this monotonicity. Comparison with previous flows obtained along null directions and on initial data sets is made.

1 Brief introduction to the Penrose inequality.

The Penrose inequality is a conjecture stating that, in an asymptotically flat spacetime satisfying the dominant energy condition, the total ADM mass M and the area $|S|$ of an "outermost" apparent horizon S must satisfy

$$M \geq \sqrt{\frac{|S|}{16\pi}}. \tag{1}$$

A similar inequality is conjectured for the Bondi mass. Penrose [1] gave a heuristic argument for this inequality based on the "standard view" of gravitational collapse, involving cosmic censorship, a final (nearly) stationary configuration and the classification of stationary black holes, among others.

A rigorous proof, however, has been achieved only in two particular cases: spherical symmetry [2, 3] and the time-symmetric case. The latter can be formulated as a purely Riemannian problem, termed *Riemannian Penrose inequality* which involves a Riemannian, asymptotically Euclidean 3-manifold (Σ, γ), with non-negative Ricci scalar (which translates the dominant energy condition) and with an outermost minimal surface S (which substitutes the apparent horizon). Inequality (1) for this case was proven by Huisken and Ilmanen [4] for connected S and by Bray [5], with a totally different method, for arbitrary S. Huisken and Ilmanen's approach makes use of a special flow of two-surfaces called Inverse Mean Curvature Flow (IMCF). Bray's method uses a flow of Riemannian metrics interpolating between the starting metric γ and the metric of a $t =$ const. slice of Schwarzschild outside the horizon. This flow of metrics keeps the area of the outermost minimal surface constant and does not increase the ADM mass. The Penrose inequality follows from the fact that equality holds for Schwarzschild.

The IMCF method is based on an observation due to Geroch [6] that a certain quantity, which we call Geroch-Hawking quasi-local mass, associated to any closed surface is non-decreasing along such flow. This quantity approaches the ADM mass for suitable surfaces at infinity (such as metric 2-spheres) and equals the right hand side of (1) at the minimal surface. Thus, if the IMCF exists globally and approaches infinitely large round spheres, the Penrose inequality follows. Ignoring existence and regularity issues, this "proof" was first given by Jang and Wald [7]. However, the IMCF generically develops singularities, so the issue remained open. Huisken and Ilmanen [4] succeeded in defining a suitable weak version of the flow for which global existence could be proven. This weak version of the IMCF allows for jumps and one needs to worry about monotonicity of the Geroch-Hawking mass at the jumps. The condition of connectedness of the starting minimal surface was required at this point.

The Geroch-Hawking mass is a translation to 3-dimensional Riemannian manifolds of a similar object defined for surfaces in spacetime, the Hawking mass [8]. This object also approaches the ADM for suitable large surfaces and gives the right hand side of (1) on any apparent horizon. Thus, in principle it is possible that the Hawking mass may also play an important role in proving the general Penrose inequality. Monotonicity of the Hawking mass along null directions was analyzed by Hayward [3]. More recently, the rate of change of the Hawking mass along an IMCF in an arbitrary initial data set (Σ, g_{ij}, k_{ij}) was studied by Malec, Mars and Simon [9]. In the present contribution we determine the change of the Hawking mass under an *arbitrary* flow in spacetime and analyze under which circumstances monotonicity is achieved. This spacetime approach, in particular, allows to unify the null and spacelike approaches mentioned above and will, hopefully, improve our general understanding of monotonicity of the Hawking mass.

2 Variation of the Hawking mass under arbitrary flows.

The Hawking mass is a scalar functional defined on the set of C^1 two-dimensional orientable closed (i.e. compact without boundary) spacelike surfaces S embedded in a spacetime (V, g). Its induced metric will be denoted by h, and its extrinsic curvature vector by $\vec{K}$. The latter is a symmetric tensor which maps pairs of vectors tangent to S to a normal vector at the same point. The defining formula, which at the same time fixes our sign conventions, is

$$\nabla_{\vec{X}}\vec{Y} = \overline{\nabla}_{\vec{X}}\vec{Y} - \vec{K}\left(\vec{X},\vec{Y}\right), \tag{2}$$

where $\vec{X},\vec{Y}$ are tangent vector fields on S and $\overline{\nabla}$ is the Levi-Civita connection on (S,h). The two terms in the right hand side correspond to the decomposition $T_pV = T_pS \oplus N_pS$ into tangent and orthogonal components of a vector. Here T_pS and N_pS denote the tangent and normal spaces to S at p. According to this decomposition we shall write $\vec{V} = \vec{V}^{\parallel} + \vec{V}^{\perp}$ for any vector $\vec{V} \in T_pV$. The collection of all normal spaces of S organize themselves into a vector bundle called *normal bundle*, which can be canonically endowed with a connection $\nabla^{\perp}$ defined by

$$\nabla^{\perp}_{\vec{X}}\vec{\xi} \equiv \left(\nabla_{\vec{X}}\vec{\xi}\right)^{\perp},$$

where $\vec{\xi}$ is any normal vector field to S. Being (S,h) orientable, it has a canonical two-form $\boldsymbol{\eta_S}$, unique except for a sign. Fixing this sign amounts to fixing an orientation on S, which we do. This, combined with the spacetime volume form, gives an orientation to N_pS, which allows us to define a Hodge dual operation on normal vector fields. Explicitly

$$V^{\star\alpha} = \frac{1}{2}\eta^{\alpha}{}_{\beta\gamma\delta}V^{\beta}e^{\gamma}_{A}e^{\delta}_{B}\eta_S^{AB},$$

where $\{\vec{e}_A\}$ is an oriented basis of the tangent space and A, B are indices of tensors in S. Immediate properties of the Hodge dual operation are $\vec{V}^{\star\star} = \vec{V}$ and $(\vec{V}\cdot\vec{U}^{\star}) = -(\vec{V}^{\star}\cdot\vec{U})$, where $(\cdot)$ indicates scalar product with the spacetime metric. Therefore any normal vector is orthogonal to its dual and both have opposite norms. In particular, a normal null vector is either self-dual, $\vec{l}^{\star} = \vec{l}$ or anti-self dual $\vec{l}^{\star} = -\vec{l}$. Generically we will denote the former by $\vec{l}^{+}$ and the latter by $\vec{l}^{-}$.

A fundamental object in the following is the mean curvature vector $\vec{H} = \mathrm{tr}_h\vec{K}$ of the surface. We also need the concept of trapped surface, which was put forward by Penrose and extensively used in several topics, including the singularity theorems. Penrose's definition was written in terms of expansions of null orthogonal congruences. They can be rewritten in terms of $\vec{H}$. ¿From now on S will denote a C^1 closed, oriented spacelike embedded 2-dimensional submanifold of (V,g). We have the following definition (see also [10] for a more refined version)[1].

Definition 1 *A surface S is called*

- *a* **future trapped surface** *if $\vec{H}$ is timelike and future directed everywhere on S.*
- *an* **absolutely non-trapped surface** *if $\vec{H}$ is spacelike everywhere on S.*

[1] Our concept of marginally trapped surface is slightly more general than the original one [11] as $\vec{H}$ could swap between the two null orthogonal directions passing through the zero vector (something not allowed in the standard definition). We remark that, as usual, $\vec{0}$ is null but not timelike or spacelike.

- *a* **future marginally trapped surface** *if $\vec{H}$ is null and future directed everywhere on S.*

And corresponding definitions for past marginally trapped or past trapped surfaces. Let us now recall the definition of the Hawking quasi-local mass [8]

$$M_H(S) = \sqrt{\frac{|S|}{16\pi}}\left(\frac{\chi(S)}{2} - \frac{1}{16\pi}\int_S \vec{H}^2 \boldsymbol{\eta_S}\right), \tag{3}$$

where $\chi(S)$ is the Euler characteristic of S and $\vec{H}^2 = (\vec{H}\cdot\vec{H})$. For surfaces embedded in a three-dimensional manifold the mean curvature vector $\vec{H}$ is obviously aligned to the unique normal direction of S, $\vec{\nu}$. We thus have $\vec{H} = p\vec{\nu}$. The Hawking-Geroch mass is defined exactly as (3) with $\vec{H}^2 \to p^2$. Notice that a surface embedded in a hypersurface Σ in a spacetime will generally have different Hawking and Geroch-Hawking masses.

Our aim is to calculate the variation of the Hawking mass along an arbitrary flow in spacetime. Thus, let us fix one arbitrary C^2 surface S among the class we are considering (the extra degree of differentiability is needed to perform the variation) and take an arbitrary embedding

$$\begin{aligned} \Phi : \hat{S} \times I &\longrightarrow V, \\ (x,\lambda) &\longrightarrow \Phi(x,\lambda) \equiv \Phi_\lambda(x), \end{aligned}$$

where $\hat{S}$ is an abstract manifold diffeomorphic to S and $I \ni 0$ is an open interval of $\mathbb{R}$. No restriction is imposed on Φ except of being C^1, with C^2 spacelike leaves $S_\lambda \equiv \Phi_\lambda(\hat{S})$ and $S_0 = S$. In particular the embedded hypersurface $\mathcal{N} \equiv \Phi(\hat{S}\times I)$ is not assumed to be spacelike and its tangent space may change between timelike, spacelike or null at different points. The flow of two-surfaces S_λ has a flow vector $\vec{V} = d\Phi(\partial_\lambda)$ which may have an arbitrary causal character. It is convenient to define $\vec{\xi}$ as its normal component with respect to S_λ, $\vec{\xi} = \vec{V}^\perp$. For simplicity we shall still call this vector "flow vector" (or also "velocity vector").

¿From the flow of surfaces a function $M_H(\lambda) \equiv M_H(S_\lambda)$ may be defined. Our aim is to evaluate its derivative with respect to λ. This will involve, in particular, the λ-derivative of the area of the surfaces. This is the well-known first variation of area, which reads (see e.g. [12])

$$\frac{d\,|S_\lambda|}{d\lambda} = \int_{S_\lambda} \left(\vec{\xi}\cdot\vec{H}\right)\boldsymbol{\eta_{S_\lambda}}.$$

For later use it is convenient to define, on each leaf, the mean value of $(\vec{\xi}\cdot\vec{H})$. This defines a function of λ

$$g(\lambda) \equiv \frac{\int_{S_\lambda}\left(\vec{\xi}\cdot\vec{H}\right)\boldsymbol{\eta_{S_\lambda}}}{|S_\lambda|},$$

in terms of which we obviously have $d|S_\lambda|/d\lambda = g(\lambda)|S_\lambda|$. Regarding the λ-derivative of $\int_{S_\lambda}\vec{H}^2\boldsymbol{\eta_{S_\lambda}}$, the details of the derivation are rather long and will be omitted here. The final result is

Lemma 1 *Let S and Φ be defined as before. The derivative of the Hawking mass along the flow reads*

$$\frac{dM_H(S_\lambda)}{d\lambda} = \frac{1}{16\pi}\sqrt{\frac{|S_\lambda|}{16\pi}}\int_{S_\lambda}\left[2\left(\vec{\Pi}_{AB}\cdot\vec{H}\right)\left(\vec{\Pi}^{AB}\cdot\vec{\xi}\right) - \left(\vec{\Pi}_{AB}\cdot\vec{\Pi}^{AB}\right)\left(\vec{H}\cdot\vec{\xi}\right) + \right.$$
$$\left. 2G_{\alpha\beta}H^{\star\,\alpha}\xi^{\star\,\beta} + \left[\left(\vec{H}\cdot\vec{\xi}\right) - g(\lambda)\right]\left(\vec{H}^2 - R(h_\lambda)\right) + 2tr_{S_\lambda}\left(\vec{H}\cdot\nabla^{\perp}\nabla^{\perp}\vec{\xi}\right)\right]\boldsymbol{\eta_{S_\lambda}}, \quad (4)$$

where $\vec{\Pi}$ is the trace-free part of the extrinsic curvature vector, $\vec{\Pi} = \vec{K} - \frac{1}{2}\vec{H}h$ and $R(h_\lambda)$ is the Ricci scalar of (S_λ, h_λ).

Remark. The proof of this lemma uses in a non-trivial way that both the dimension of S_λ and its codimension are 2. The result, as it stands, holds only for surfaces in a (4-dimensional) spacetime.

3 Monotonicity properties of the Hawking mass.

In order to discuss flow conditions under which the Hawking mass is monotone, it is convenient to define four closed subsets of the normal space N_pS_λ. As we just remarked, this is a two-dimensional Lorentzian space and we can speak of causal (i.e. timelike or null) vectors and of achronal (i.e. spacelike or null) vectors. Let us denote by $C_p^{\pm}$ the subset of future (past) causal vector and $A_p^{\pm}$ the set of vectors such that its Hodge dual is future (past) causal. Notice that all the vectors in $A_p^{\pm}$ are achronal. These four sets clearly cover the normal space N_pS_λ and they are not disjoint. The $\vec{0}$ vector belongs to all of them. Furthermore, the intersection $C^+\cap A^+$ (we drop the subindex p from now on for simplicity) is the set of self-dual future null vectors (and similarly for $C^-\cap A^-$). As for $C^+\cap A^-$, this is the set of anti-self dual future null vectors (and past for $C^-\cap A^+$). This decomposition can be easily visualized in the figure below

We can now discuss restrictions under which each term in (4) is non-negative. Again, we shall not enter into the proofs and will only present the results. Let us start with the first two terms within the integral sign in (4). Clearly, both have the same nature and need to be treated together. This analysis is easy as it only involves algebraic manipulations. We find

Lemma 2 *The inequality*

$$2\left(\vec{\Pi}_{AB}\cdot\vec{H}\right)\left(\vec{\Pi}^{AB}\cdot\vec{\xi}\right) - \left(\vec{\Pi}_{AB}\cdot\vec{\Pi}^{AB}\right)\left(\vec{H}\cdot\vec{\xi}\right) \geq 0$$

holds for any trace-free symmetric tensor $\vec{\Pi}_{AB}$ if and only if $\vec{\xi},\vec{H}\in C^+$, $\vec{\xi},\vec{H}\in C^-$, $\vec{\xi},\vec{H}\in A^+$ or $\vec{\xi},\vec{H}\in A^-$,

In short, the first two terms of $\frac{dM_H(\lambda)}{d\lambda}$ are monotone iff the velocity vector of the flow points into the same quadrant as the mean curvature vector, at each point on the surface.

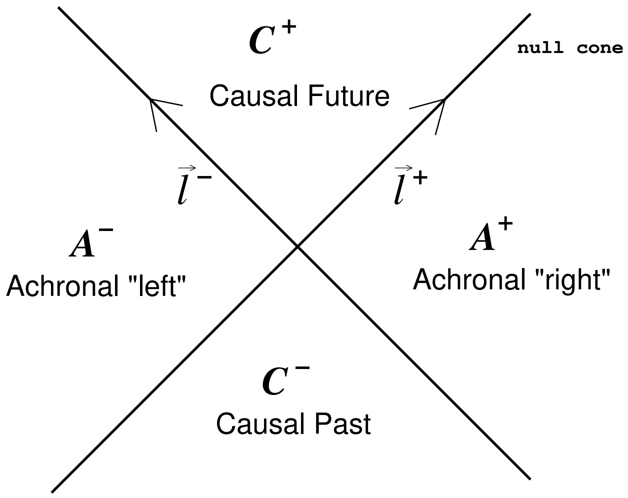

Figure 1: Decomposition of the normal space into four closed sets.

Analyzing the second term $2G_{\alpha\beta}H^{\star\,\alpha}\xi^{\star\,\beta}$ is trivial. This expression involves the Einstein tensor of the spacetime, so in order to ensure its non-negativity some condition on the spacetime is needed. As already mentioned, the Penrose inequality is expected to hold on spacetimes satisfying the dominant energy condition (DEC), i.e. $G_{\alpha\beta}u^\alpha v^\beta \geq 0$ for any pair of future (or past) causal vectors. In this setting, $2G_{\alpha\beta}H^{\star\,\alpha}\xi^{\star\,\beta} \geq 0$ holds provided both $\vec{H}^\star$ and $\vec{\xi}^\star$ are future (past) causal, or, equivalently $\vec{\xi}, \vec{H} \in A^+$ or $\vec{\xi}, \vec{H} \in A^-$. Thus the flows for which the Hawking mass is monotone are generically restricted to having achronal velocities and leaves with achronal mean curvature, both pointing into the same quadrant.

The third term to deal with is a product of two factors

$$\left[\left(\vec{H}\cdot\vec{\xi}\right) - g(\lambda)\right]\left(\vec{H}^2 - R(h_\lambda)\right). \tag{5}$$

This object will be non-negative provided each factor has the same sign everywhere. However, the first factor is the difference between an object, $(\vec{H}\cdot\vec{\xi})$, and its mean value on S_λ, $g(\lambda)$. Consequently, it will *never* have a constant sign on the surface unless it is identically zero. Moreover, we have no a priori control on the sign of $\vec{H}^2 - R(h_\lambda)$, so this term cannot be expected to change sign exactly at those places where the first factor does (except by choosing the velocity vector precisely in such a way that this happens). The only "natural" way of getting non-negativity (in fact vanishing) of (5) is by imposing

$$\left(\vec{\xi}\cdot\vec{H}\right) = g(\lambda). \tag{6}$$

i.e. constancy of the scalar product on each leaf. There is no restriction a priori on the value of this constant on each leaf of the foliation. However, the geometric flow of 2-surfaces is obviously independent of the parametrization used, and a reparametrization changes the velocity field $\vec{\xi}$ by a (nowhere zero) constant on each leaf. Such reparametrization would rescale $g(\lambda)$ by an arbitrary (never zero) function of λ. Thus, if $g(\lambda) \neq 0$ everywhere, $g = 1$ can be assumed without loss of generality. If, on the other hand, g has zeros (be it at isolated points or on intervals), those values are meaningful and independent of reparametrization.

Condition (6) is the generalization to codimension two of inverse mean curvature flow for surfaces in a 3-dimensional space. Indeed, in codimension one, the mean curvature vector and the normal velocity are obviously parallel, and the condition $(\vec{H} \cdot \vec{\xi}) = 1$ just states $\vec{\xi} = \vec{\nu}/p$, where $\vec{\nu}$ is the unit normal vector (and p the mean curvature, as before). This is precisely the IMCF condition. Notice, however, that IMCF in codimension one fixes uniquely the velocity of the flow while (6) still leaves room for imposing one extra scalar condition on the velocity. Nevertheless, we shall keep the name IMCF for (6).

Finally we need to consider the fourth term $2\mathrm{tr}_{S_\lambda}\left(\vec{H} \cdot \nabla^\perp \nabla^\perp \vec{\xi}\right)$. This object is dealt with most easily once we assume that (6) holds. The following lemma gives *sufficient* conditions for the nonnegativity of this term. We first note that if $\vec{\xi}$ is zero on an open set of S_λ then the term above is obviously zero on that subset. So, we can restrict ourselves to velocities which are non-zero almost everywhere on S_λ.

Lemma 3 *Fix an arbitrary value of λ and suppose that the normal velocity $\vec{\xi}$ is not zero on open subsets of S_λ. Assume also that the IMCF condition $(\vec{\xi} \cdot \vec{H}) = g(\lambda)$ is satisfied. Then,*

(a) If $g(\lambda) = 0$ (i.e. $\vec{H}|_{S(\lambda)} = B\vec{\xi}^\star|_{S(\lambda)}$ for some function B on the surface) then $\int_{S_\lambda}\left[tr_{S_\lambda}\left(\vec{H} \cdot \nabla^\perp \nabla^\perp \vec{\xi}\right)\right]\boldsymbol{\eta}_{S_\lambda} \geq 0$ is implied by any one of the following two conditions: (i) $\vec{\xi}$ is null all over S_λ or (ii) $B = const.$ on S_λ.

(b) If $g(\lambda) \neq 0$, $\int_{S_\lambda}\left[tr_{S_\lambda}\left(\vec{H} \cdot \nabla^\perp \nabla^\perp \vec{\xi}\right)\right]\boldsymbol{\eta}_{S_\lambda} \geq 0$ is implied by the scalar condition $(\vec{\xi} \cdot \vec{H}^\star) = F(\lambda)$ with $F(\lambda)$ constant on S_λ and any one of the following four possibilities at each point of S_λ: (i) $\vec{\xi} \in C^+, \vec{H} \in C^-$, (ii) $\vec{\xi} \in C^-, \vec{H} \in C^+$, (iii) $\vec{\xi}, \vec{H} \in A^+$, (iv) $\vec{\xi}, \vec{H} \in A^-$.

We must remark that this lemma does not give pointwise non-negativity of the fourth term involved. Rather it gives non-negativity of the integral on the surface. The reason is that, even under the conditions of the lemma, divergence terms survive, and they only integrate to zero. Regarding case (b) in the lemma, we must stress that, in principle, it is perfectly possible that $\vec{H}$ and $\vec{\xi}$ change quadrants in the normal space N_pS_λ when the point p varies on the surface. This case is still covered by the lemma, provided one of the conditions (i),(ii),(iii) or (iv) holds at every point.

This lemma tells us that in the generic case $g(\lambda) \neq 0$, besides imposing IMCF, we need to restrict also the scalar product $(\vec{\xi} \cdot \vec{H}^\star)$ to being constant on each leaf. Among the conditions (i) to (iv) that ensure monotonicity, only (iii) and (iv) (i.e. velocity vector and mean curvature vector both achronal and to the same quadrant) give conditions which are compatible with those ensuring non-negativity of the other terms in (4).

Regarding the case of $g(\lambda) = 0$ on S_λ, the orthogonality of $\vec{H}$ and $\vec{\xi}$ implies that they cannot belong to the same quadrant (thus violating the non-negativity condition in lemma 2) unless $\vec{\xi}$ is null and $\vec{H} = B\vec{\xi}$, with a non-negative arbitrary *function* on S_λ. Thus, when the surface is marginally trapped and the flow is null, monotonicity is very easily achieved (initially), the only condition being that the velocity is pointwise parallel to the mean curvature of the surface. Of course, the property of $\vec{H}$ being null will not be maintained by the flow and the flow vector will have to adjust (if at all possible) to keep a monotone flow.

Summarizing all the results above, we can write down the following theorem which gives sufficient conditions on a spacetime flow of 2-surfaces so that the Hawking mass is monotone along the flow.

Theorem 1 *Let S be an oriented, spacelike, closed, C^2 embedded 2-surface in a spacetime (V, g). Let $\vec{\xi}$ be a C^2 normal vector field on S, $\vec{H}$ the mean curvature vector of the surface and $M_H(S)$ its Hawking mass. Let S_λ be any flow of spacelike two-surfaces starting at S with $\vec{\xi}$ as normal component of the initial velocity. Then, $\frac{dM_H(S_\lambda)}{d\lambda}|_{\lambda=0} \geq 0$ holds whenever the following four conditions hold*

(1) The spacetime satisfies the dominant energy condition.

(2) The mean curvature vector $\vec{H}$ is spacelike or null on S, and $\vec{\xi}$ points into the same quadrant as $\vec{H}$.

(3) The IMCF condition holds, i.e. $(\vec{\xi} \cdot \vec{H}) = g_0$, where g_0 is a non-negative constant.

(4) The dual IMCF condition also holds, i.e. $(\vec{\xi} \cdot \vec{H}^\star) = F_0$, where F_0 is a constant satisfying the bound $|F_0| \leq g_0$.

This is the main result of the contribution. We shall conclude by discussing several particular cases where the conditions of the theorem are fulfilled and comparing them to previously known monotonicity results.

The first situation involves surfaces where the flow velocity is null everywhere, while the mean curvature vector of the surface is achronal everywhere. Only one condition needs to be imposed in this case, namely $(\vec{\xi} \cdot \vec{H}) = 1$. Indeed, using that $\vec{\xi}^\star = \pm\vec{\xi}$ for null vectors, we get

$$\left(\vec{\xi} \cdot \vec{H}^\star\right) = -\left(\vec{\xi}^\star \cdot \vec{H}\right) = \mp\left(\vec{\xi} \cdot \vec{H}\right) = \mp 1,$$

so that $g_0 = 1$ and $F_0 = \mp 1$ and all the conditions of the theorem hold (provided $\vec{\xi}$ and $\vec{H}$ point into the same quadrant). Notice that the IMCF condition $(\vec{\xi} \cdot \vec{H}) = 1$ prevents $\vec{\xi}$ or $\vec{H}$ from vanishing anywhere on the surface. This null monotone flow was discovered by Hayward [3] and was termed *uniformly expanding* by the author.

Another particular case for which the Hawking mass is monotone is the so-called *inverse mean curvature flow vector*. It requires $\vec{H}$ spacelike everywhere on S. The velocity vector of the flow is parallel to $\vec{H}$ and its norm is the inverse of $||\vec{H}||$, i.e. $\vec{\xi} = \frac{\vec{H}}{\vec{H}^2}$. It is immediate to see that all conditions of the theorem are fulfilled and $g_0 = 1$, $F_0 = 0$. This flow was mentioned by Huisken and Ilmanen in [4] and discussed later by Frauendiener [13].

A third flow with monotone Hawking mass involves surfaces S with $\vec{H}$ spacelike everywhere. The velocity flow contains one arbitrary function of one variable $F(\lambda)$ satisfying $|F(\lambda)| \leq 1$. The velocity of the flow reads

$$\vec{\xi} = \frac{1}{\vec{H}^2} \left(\vec{H} - F(\lambda)\vec{H}^{\star} \right). \tag{7}$$

In the particular case $|F(\lambda)| < 1$, this is the spacetime version of the flows found by Malec, Mars and Simon [9] in terms of initial data sets. Indeed, in this case the vector $\vec{\xi}$ in (7) is spacelike by construction, so that the hypersurface Σ generated by the flow of 2-surfaces will be spacelike. We can therefore translate the geometry of S_λ as spacetime submanifolds in terms of their geometry as submanifolds of Σ together with the spacetime geometry of this hypersurface. Thus, the flow can be viewed as a flow of 2-surfaces in a 3-dimensional Riemannian manifold. The IMCF condition $(\vec{\xi} \cdot \vec{H}) = 1$ can be easily seen to translate into the condition that S flows within Σ as an IMCF. The other condition $(\vec{\xi} \cdot \vec{H}^{\star}) = F$ becomes a condition involving p (the mean curvature of $S \subset \Sigma$) and q, the trace on S of the extrinsic curvature of Σ as a hypersurface in spacetime, which explicitly reads $q/p = F(\lambda)$. This is precisely the condition obtained in [9] (see also [14]) in order to ensure monotonicity of the Hawking mass in an initial data setting.

Finally, a fourth flow guarantying monotonicity of the Hawking mass involves surfaces with null $\vec{H}$ and velocity vector which is either spacelike or null (towards the same quadrant as $\vec{H}$) and satisfies $(\vec{\xi} \cdot \vec{H}) = 1$. Thus, the surface S is restricted to being a future (past) marginally trapped surface, which is precisely the starting surface for flows aiming at proving the Penrose inequality. Given $\vec{H}$, the velocity vector $\vec{\xi}$ at S has a freedom of one arbitrary *function*. Thus, we are allowed to flow out of a marginally trapped surface along a large class of flows for which M_H is monotone. It is worth emphasizing that this, and all the other explicit flows discussed in this final part, can be obtained as suitable limits of (7), which therefore should be considered as specially relevant.

A final comment is in order. The work presented here analyzes, from a spacetime point of view, flows of surfaces for which the Hawking mass is monotone. The use of a spacetime analysis has allowed us to unify and clarify the relationships between previous flows presented in the literature for which M_H is monotone. As mentioned in the introduction, our main aim in performing this analysis points towards an eventual proof of the Penrose inequality in the general case, i.e. for general horizons. We must emphasize however, that obtaining a flow with monotone Hawking mass is just the very first step in building a proof along the lines followed by Huisken and Ilmanen. Indeed, not even local existence is known for the flow (7), let alone a weak formulation which could allow the flow to exist globally. Nevertheless the existence of spacetime flows for which the Hawking mass is monotone is remarkable in itself and, I believe, it deserves further careful investigation.

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