The ADM canonical approach to the problem of motion of binary black holes

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Outline

- ADM Hamiltonian and Routh Functional
- Brill-Lindquist vs Misner-Lindquist BHs
- Contact Transformation
- Post-Newtonian Expansions
- 3PN Binary BH Conservative Dynamics
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ADM Hamiltonian and Routh Functional

$$g^{1/2}\mathrm{R} = \frac{1}{g^{1/2}}\left(\pi^i_j\pi^j_i - \frac{1}{2}\pi^i_i\pi^j_j\right) + \frac{16\pi G}{c^3}\sum_a \left(m_a^2c^2 + g^{ij}p_{ai}p_{aj}\right)^{1/2}\delta_a$$

$$-2\partial_j\pi^j_i + \pi^{kl}\partial_i g_{kl} = \frac{16\pi G}{c^3}\sum_a p_{ai}\delta_a$$

3 Cordinate Conditions: $g_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h^{\mathrm{TT}}_{ij}$

CC: $\pi^{ii} = 0, \qquad \pi^{ij} = -g^{1/2}(K^{ij} - g^{ij}K), \quad \pi^i_i = \pi^{ij}h^{\mathrm{TT}}_{ij}$

unique decomposition: $\pi^{ij} = \tilde{\pi}^{ij} + \pi^{ij}_{\mathrm{TT}}$

$$\tilde{\pi}^{ij} = \partial_i \pi^j + \partial_j \pi^i - \frac{2}{3}\delta_{ij}\partial_k \pi^k$$

$\pi^{ij}_{\mathrm{TT}} c^3/16\pi G$: canonical conjugate to h^{TT}_{ij}

ADM Hamiltonian

$$H\left[x^i_a, p_{ai}, h^{\mathrm{TT}}_{ij}, \pi^{ij}_{\mathrm{TT}}\right] = -\frac{c^4}{16\pi G}\int d^3x \; \Delta\phi\left[x^i_a, p_{ai}, h^{\mathrm{TT}}_{ij}, \pi^{ij}_{\mathrm{TT}}\right]$$

Routh functional

$$R\left[x^i_a, p_{ai}, h^{\mathrm{TT}}_{ij}, \partial_t h^{\mathrm{TT}}_{ij}\right] = H - \frac{c^3}{16\pi G}\int d^3x \; \pi^{ij}_{\mathrm{TT}}\partial_t h^{\mathrm{TT}}_{ij}$$

$$\frac{\delta \int R(t')dt'}{\delta h^{\mathrm{TT}}_{ij}(x^k, t)} = 0, \quad \dot{p}_{ai} = -\frac{\partial R}{\partial x^i_a}, \quad \dot{x}^i_a = \frac{\partial R}{\partial p_{ai}}$$

conservative Routh functional:

$$R_c(t) = \frac{1}{2}[R_{on}(t) + R_{on}(-t)]$$

on-field-shell Routh functional:

$$R_{on}(t) = R\left[x_a^i, p_{ai}, h_{ij}^{\mathrm{TT}}[x_a^k, p_{ak}], \partial_t h_{ij}^{\mathrm{TT}}[x_a^k, p_{ak}]\right]$$

$$\dot{p}_{ai}(t) = -\frac{\delta \int R_c(t')dt'}{\delta x_a^i(t)}, \qquad \dot{x}_a^i(t) = \frac{\delta \int R_c(t')dt'}{\delta p_{ai}(t)}$$

$$\frac{\delta \int R_c(t')dt'}{\delta z(t)} = \frac{\partial R_c}{\partial z(t)} - \frac{d}{dt}\frac{\partial R_c}{\partial \dot{z}(t)} + \ldots, \qquad z = (x_a^i, p_{ai})$$

Brill-Lindquist vs Misner-Lindquist BHs

$$-\left(1+\frac{1}{8}\,\phi\right)\Delta\phi=\frac{16\pi G}{c^2}\sum_a m_a\delta_a \qquad (h_{ij}^{\mathrm{TT}}=0=p_{ai})$$

$$\phi=\frac{4G}{c^2}\left(\frac{\alpha_1}{r_1}+\frac{\alpha_2}{r_2}\right)$$ unique solution, see P.J.'s talk

$$\alpha_a=m_a-\frac{m_a+m_b}{2}+\frac{c^2 r_{ab}}{G}\left(\sqrt{1+\frac{m_a+m_b}{c^2 r_{ab}/G}+\left(\frac{m_a-m_b}{2c^2 r_{ab}/G}\right)^2}-1\right)$$

$$H_{\mathrm{BL}}=(\alpha_1+\alpha_2)\;c^2=(m_1+m_2)\;c^2-G\;\frac{\alpha_1\alpha_2}{r_{12}}$$

$$H^{3PN}_{\text{ambiguous}} = \omega_{\text{static}}(m_1 + m_2)\frac{G^4 m_1^2 m_2^2}{c^6 r_{12}^4} = \omega_{\text{static}}\frac{G^4 M^3 \mu^2}{c^6 r_{12}^4}$$

$\omega_{\text{static}} = 0$ BL, $\omega_{\text{static}} = -\frac{1}{8}$ ML (shift of positions

$$H^{3PN}_{\text{induced}} = -\omega_{\text{static}}\left(\mathbf{p}^2 - 3(\mathbf{n}\cdot\mathbf{p})^2\right)\frac{G^3 M^2}{c^6 r_{12}^3}$$ relative to BL)

Thus, coordinate (gauge) transformation (given in CMS):

$$\delta H = \omega_{\text{static}}\left(\mathbf{p}^2 - 3(\mathbf{n}\cdot\mathbf{p})^2\right)\frac{G^3 M^2}{c^6 r_{12}^3} - \omega_{\text{static}}\frac{G^4 M^3 \mu^2}{c^6 r_{12}^4}$$

Conformal-flat assumption only ($h_{ij}^{\mathrm{TT}}=0$). Then:

No solutions of the constraint equations by dim. reg. (see P.J.'s talk)

$$\Delta\psi = -\frac{1}{8}\pi^i_j\pi^j_i\frac{1}{\psi^7} - \frac{2\pi G}{c^2}\sum_a m_a\delta_a\frac{1}{\psi}(1+\frac{p_a^2}{m_a^2\psi^4})^{1/2}$$

$$\partial_j\pi^j_i = -\frac{8\pi G}{c^3}\sum_a p_{ai}\delta_a, \qquad \psi\equiv 1+\frac{\phi}{8}$$

Particularly puncture-type methods (see below) are not well defined; but skeleton truncation is (see G.F.'s talk).

$$\psi = 1+\frac{G}{2c^2}(\frac{\alpha_1}{r_1}+\frac{\alpha_2}{r_2})+u$$

$$\psi_a \alpha_a = m_a (1 + \frac{p_a^2}{m_a^2 \psi_a^4})^{1/2} \tag{1}$$

$$\Delta u = -\frac{1}{8} \pi^i_j \pi^j_i \frac{1}{\psi^7} \tag{2}$$

$$\psi_a = 1 + \frac{G}{2c^2} \sum_{b \neq a} \frac{\alpha_b}{r_{ab}} + u_a \tag{3}$$

The regularization as in (3), used in (1), is incompatible with (2).

$$H_{\mathrm{ADM}} = -\frac{c^4}{2\pi G} \int d^3x \Delta\psi$$

$$= \sum_a c^2 \frac{m_a}{\psi_a} (1 + \frac{p_a^2}{m_a^2 \psi_a^4})^{1/2} + \frac{c^4}{16\pi G} \int d^3x \pi^i_j \pi^j_i \frac{1}{\psi^7}$$

Contact Transformation

conservative ordinary Hamiltonian $\bar{H}_{co}(\bar{x}^i_a, \bar{p}_{ai})$:

$$\dot{\bar{p}}_{ai} = -\frac{\partial \bar{H}_{co}}{\partial \bar{x}^i_a}, \qquad \dot{\bar{x}}^i_a = \frac{\partial \bar{H}_{co}}{\partial \bar{p}_{ai}}$$

$$\bar{H}_{co}(x,p) \equiv R_c(x,p,\dot{x}(x,p),\dot{p}(x,p)), \quad L_c = p_{ai}\dot{x}^i_a - R_c$$

$$\begin{aligned} L_c(x,p,\dot{x},\dot{p}) &= \bar{L}_{co}(x,p) + \frac{\delta L_c}{\delta x^i_a}\frac{\partial R_c}{\partial \dot{p}_{ai}} - \frac{\delta L_c}{\delta p_{ai}}\frac{\partial R_c}{\partial \dot{x}^i_a} + \text{Double Zero} \\ &= \bar{L}_{co}(\bar{x},\bar{p}) + DZ, \quad \bar{x}^i_a = x^i_a + \delta x^i_a,\ \bar{p}_{ai} = p_{ai} + \delta p_{ai} \end{aligned}$$

$$\delta x^i_a = \frac{\partial R_c}{\partial \dot{p}_{ai}}, \quad \delta p_{ai} = -\frac{\partial R_c}{\partial \dot{x}^i_a}$$

Post-Newtonian Expansions

$$R\left[x_a^i, p_{ai}, h_{ij}^{\mathrm{TT}}, \partial_t h_{ij}^{\mathrm{TT}}\right] - Mc^2 = \sum_{n=0}^{\infty} \left(\frac{1}{c^2}\right)^n R_n \left[x_a^i, p_{ai}, \hat{h}_{ij}^{\mathrm{TT}}, \partial_t \hat{h}_{ij}^{\mathrm{TT}}\right]$$

$$h_{ij}^{\mathrm{TT}} = \tfrac{G}{c^4} \hat{h}_{ij}^{\mathrm{TT}}$$

$$\left(\Delta - \frac{\partial_t^2}{c^2}\right) h^{\mathrm{TT}} = \frac{G}{c^4} \sum_{n=0}^{\infty} \left(\frac{1}{c^2}\right)^n D_n^{\mathrm{TT}}[x, x_a(t), p_a(t), \hat{h}^{\mathrm{TT}}(t), \partial_t \hat{h}^{\mathrm{TT}}(t)]$$

$H = H(\mathbf{p}, \mathbf{r}), \qquad p^2 = p_r^2 + j^2/r^2, \qquad p_r = (\mathbf{p}\cdot\mathbf{r})/r$

circular orbits: $p_r = 0, \qquad p^2 = j^2/r^2, \qquad H = H(j,r)$

circular motion: $\frac{\partial}{\partial r} H(j,r) = 0 \to H(j)$

orbital frequency: $\omega = \frac{dH(j)}{dj} \to j(\omega) \to H(\omega)$

ISCO: $\frac{dH(\omega)}{d\omega} = 0, \qquad \frac{dj(\omega)}{d\omega} = 0 \qquad$ [alternatively $\frac{\partial^2}{\partial r^2} H(j,r) = 0$]

$$\text{SBH:} \quad E(x) = \frac{1-2x}{(1-3x)^{1/2}} - 1$$

$$= -\frac{1}{2}x + \frac{3}{8}x^2 + \frac{27}{16}x^3 + \frac{675}{128}x^4 + \frac{3969}{256}x^5 + \ldots$$

$$E(x) \equiv \frac{H(x) - mc^2}{mc^2}, \qquad x = \left(\frac{GM\omega}{c^3}\right)^{2/3}$$

3PN Binary BH Conservative Dynamics

$$
\begin{aligned}
H(t) &= m_1 c^2 + m_2 c^2 + H_N + \frac{1}{c^2} H_{[1PN]} \\
&+ \frac{1}{c^4} H_{[2PN]} + \frac{1}{c^6} H_{[3PN]} + \ldots \\
&+ \frac{1}{c^5} H_{[2.5PN]}(t) + \frac{1}{c^7} H_{[3.5PN]}(t) + \ldots
\end{aligned}
$$

$\hat{H} = (H - Mc^2)/\mu, \qquad \mu = m_1 m_2 / M, \qquad M = m_1 + m_2$

$\nu = \mu/M, \qquad 0 \le \nu \le 1/4$

test-body case: $\nu = 0$, equal-mass case: $\nu = 1/4$

CMS: $\mathbf{p}_1 + \mathbf{p}_2 = 0, \qquad \mathbf{p} \equiv \mathbf{p}_1/\mu,$

$p_r = (\mathbf{n} \cdot \mathbf{p}), \qquad \mathbf{q} \equiv (\mathbf{x}_1 - \mathbf{x}_2)/GM, \qquad \mathbf{n} = \mathbf{q}/|\mathbf{q}|$

$$\hat{H}_N = \frac{p^2}{2} - \frac{1}{q}$$

$$\hat{H}_{[1PN]} = \frac{1}{8}(3\nu - 1)p^4 - \frac{1}{2}[(3+\nu)p^2 + \nu p_r^2]\frac{1}{q} + \frac{1}{2q^2}$$

$$\begin{aligned}
\hat{H}_{[2PN]} &= \frac{1}{16}(1 - 5\nu + 5\nu^2)p^6 \\
&+ \frac{1}{8}[(5 - 20\nu - 3\nu^2)p^4 - 2\nu^2 p_r^2 p^2 - 3\nu^2 p_r^4]\frac{1}{q} \\
&+ \frac{1}{2}[(5 + 8\nu)p^2 + 3\nu p_r^2]\frac{1}{q^2} - \frac{1}{4}(1 + 3\nu)\frac{1}{q^3}
\end{aligned}$$

$$
\begin{aligned}
\hat{H}_{[3PN]} &= \frac{1}{128}(-5+35\nu-70\nu^2+35\nu^3)p^8 \\
&+ \frac{1}{16}[(-7+42\nu-53\nu^2-5\nu^3)p^6+(2-3\nu)\nu^2 p_r^2 p^4 \\
&+ 3(1-\nu)\nu^2 p_r^4 p^2 - 5\nu^3 p_r^6]\frac{1}{q} \\
&+ [\frac{1}{16}(-27+136\nu+109\nu^2)p^4+\frac{1}{16}(17+30\nu)\nu p_r^2 p^2 \\
&+ \frac{1}{12}(5+43\nu)\nu p_r^4]\frac{1}{q^2} \\
&+ [\left(-\frac{25}{8}+\left(\frac{1}{64}\pi^2-\frac{335}{48}\right)\nu-\frac{23}{8}\nu^2\right)p^2 \\
&+ \left(-\frac{85}{16}-\frac{3}{64}\pi^2-\frac{7}{4}\nu\right)\nu p_r^2]\frac{1}{q^3}+[\frac{1}{8}+\left(\frac{109}{12}-\frac{21}{32}\pi^2\right)\nu]\frac{1}{q^4}
\end{aligned}
$$

Dynamical Invariants

radial action $i_r(E,j)$:

$$i_r(E,j) = \frac{1}{2\pi}\oint dr\ p_r$$

phase of revolution Φ:

$$\frac{\Phi}{2\pi} = 1 + k = -\frac{\partial}{\partial j} i_r(E,j)$$

orbital period P:

$$\frac{P}{2\pi GM} = \frac{\partial}{\partial E} i_r(E,j)$$

periastron advance k:

$$
\begin{aligned}
k &= \frac{1}{c^2}\frac{3}{j^2}\left\{1+\frac{1}{c^2}\left[\frac{5}{4}(7-2\nu)\frac{1}{j^2}+\frac{1}{2}(5-2\nu)\,E\right]\right.\\
&+\left.\frac{1}{c^4}\left[a_1(\nu)\frac{1}{j^4}+a_2(\nu)\frac{E}{j^2}+a_3(\nu)\,E^2\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{P}{2\pi GM} &= \frac{1}{(-2E)^{3/2}}\left\{1-\frac{1}{c^2}\frac{1}{4}(15-\nu)E\right.\\
&+\frac{1}{c^4}\left[\frac{3}{2}(5-2\nu)\frac{(-2E)^{3/2}}{j}-\frac{3}{32}(35+30\nu+3\nu^2)\,E^2\right]\\
&+\left.\frac{1}{c^6}\left[a_2(\nu)\frac{(-2E)^{3/2}}{j^3}-3a_3(\nu)\frac{(-2E)^{5/2}}{j}+a_4(\nu)\,E^3\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
a_1(\nu) &= \frac{5}{2}\left(\frac{77}{2}+\left(\frac{41}{64}\pi^2-\frac{125}{3}\right)\nu+\frac{7}{4}\nu^2\right)\\
a_2(\nu) &= \frac{105}{2}+\left(\frac{41}{64}\pi^2-\frac{218}{3}\right)\nu+\frac{45}{6}\nu^2\\
a_3(\nu) &= \frac{1}{4}(5-5\nu+4\nu^2)\\
a_4(\nu) &= \frac{5}{128}(21-105\nu+15\nu^2+5\nu^3)
\end{aligned}
$$

Discussing ISCOs

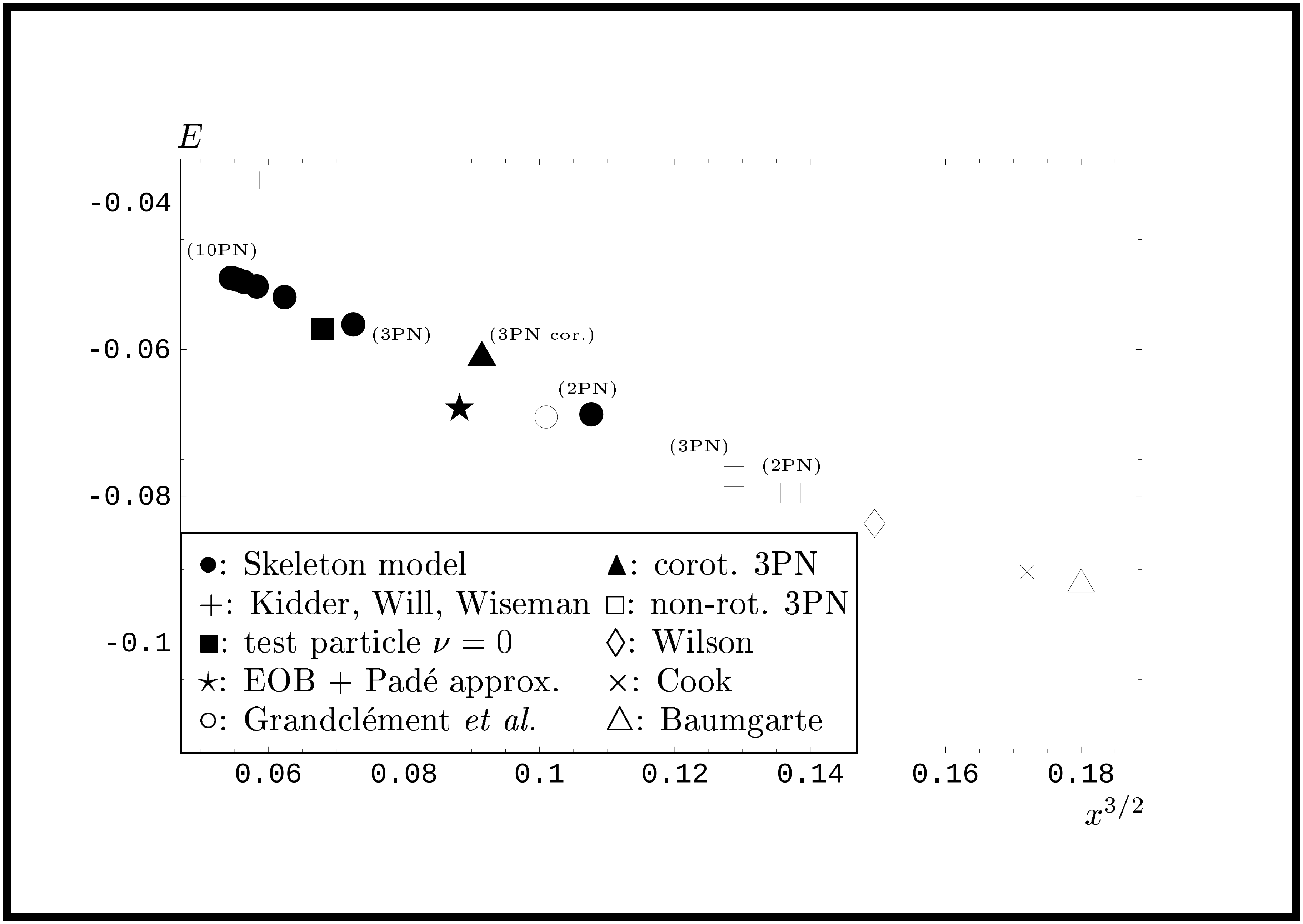
circular orbits:

$$\omega_{\rm circ} = \omega_{\rm radial} + \omega_{\rm periastron} = 2\pi \frac{1+k}{P}, \quad x = \left(\frac{GM\omega_{\rm circ}}{c^3}\right)^{2/3}$$

$$c^2 E_{3PN} \equiv \hat{H}_N + \hat{H}_{[1PN]} + \hat{H}_{[2PN]} + \hat{H}_{[3PN]}$$

$$\begin{aligned} E_{3PN}(x) &= -\frac{x}{2} + \left(\frac{3}{8} + \frac{1}{24}\nu\right) x^2 + \left(\frac{27}{16} - \frac{19}{16}\nu + \frac{1}{48}\nu^2\right) x^3 \\ &+ \left(\frac{675}{128} + \left(-\frac{34445}{1152} + \frac{205}{192}\pi^2\right)\nu + \frac{155}{192}\nu^2 + \frac{35}{10368}\nu^3\right) x^4 \end{aligned}$$

ISCO: $dE_{3PN}/dx = 0$ (representation used to compare with other authors; in approximations, ISCO is representation dependent)


E
-0.04
-0.06
-0.08
-0.1
0.06
0.08
0.1
0.12
0.14
0.16
0.18
$x^{3/2}$
(10PN)
(3PN)
(3PN cor.)
(2PN)
(3PN)
(2PN)
●: Skeleton model
+: Kidder, Will, Wiseman
■: test particle $\nu = 0$
★: EOB + Padé approx.
○: Grandclément *et al.*
▲: corot. 3PN
□: non-rot. 3PN
◇: Wilson
×: Cook
△: Baumgarte

- The difference between ‘Wilson’ (2PN, vanishing h_{ij}^{TT}) and non-rot. 2PN Einstein is just the contribution from the leading order (2PN) h_{ij}^{TT}. Higher PN order ‘Wilson’ does not exist (see page 8).
- An additional extrinsic curvature truncation yields (non-rot.) skeleton 2PN. The skeleton dynamics exists to arbitrary PN orders. But its a kind of model theory only.
- 3PN corot. vs. 3PN non-rot. (from L. Blanchet 2002) shows the influence of rigidly logged proper rotation at 3PN Einstein theory. The difference mainly results from the proper rotational energy of each object ($I\omega^2/2$) and not from the spin-orbit coupling. Corotation, when approaching ISCO, seems not very realistic.
- The Grandglément et al. (numerical, corotation, Misner-Lindquist BHs, vanishing h_{ij}^{TT}, CTS approach) value, in

regard to 3PN corot. Einstein, fits with the difference between 2PN Einstein and 2PN Wilson.

- The Cook (numerical, Misner-Lindquist BHs, vanishing h_{ij}^{TT}, CTT approach) and Baumgarte (numerical, Brill-Lindquist BHs, puncture method, vanishing h_{ij}^{TT}, CTT approach) values seem to be too strongly influenced by the 1PN dynamics which is controlled by the Bowen-York extrinsic curvature. In the bodies dynamics, the CTT and CTS approaches are identical through 2PN order inclusively (neither $\partial_t h_{ij}^{TT}$ nor π_{TT}^{ij} contribute).

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