Novel results on trapped surfaces

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October 13, 2003

Abstract

A unifying definition of *trapped submanifold* for arbitrary codimension by means of its mean curvature vector is presented. Then, the interplay between (generalized) symmetries and trapped submanifolds is studied, proving in particular that (i) stationary spacetimes cannot contain closed trapped nor marginally trapped submanifolds S of any codimension; (ii) S can be within the subset where there is a null Killing vector only if S is marginally trapped with mean curvature vector parallel to the null Killing; (iii) any submanifold orthogonal to a timelike or null Killing vector $\vec{\xi}$ has a mean curvature vector orthogonal to $\vec{\xi}$. All results are purely geometric, hold in arbitrary dimension, and can be appropriately generalized to many non-Killing vector fields, such as conformal Killing vectors and the like. A simple criterion to ascertain the trapping or not of a family of codimension-2 submanifolds is given. A path allowing to generalize the singularity theorems is conjectured as feasible and discussed.

PACS Numbers: 04.50.+h, 04.20.Cv, 04.20.Jb, 02.40.Ky

As is well-known, the concept of (closed) trapped surface, first introduced by Penrose [1], is of paramount importance in many physical problems and mathematical developments. In particular, it has been an invaluable tool for the development of the singularity theorems (see e.g. [1, 2, 3, 4]), essential for the analysis of gravitational collapse [1, 3, 5], and very useful in the study of the cosmic censorship hypothesis [6] or in the numerical evolution of apparently innocuous initial data (e.g. [7] and references therein.) This contribution is based partly on [8] and partly on [9], where more complete lists of references can be found.

Let $(\mathcal{V}, g)$ be a D-dimensional causally orientable spacetime with metric g (signature -,+,...,+), Σ any smooth orientable d-dimensional manifold, and $\Phi : \Sigma \longrightarrow \mathcal{V}$ a C^2 imbedding. Σ and its image $\Phi(\Sigma) \equiv S$ in $\mathcal{V}$ will be identified if no confusion arises. The term 'closed' is used for *compact without boundary* submanifolds. The push-forward Φ' sends a basis of $T(\Sigma)$ to a set of d linearly independent vector fields $\vec{e}_A$ on S $(A, B, \ldots = D-d, \ldots, D-1)$ which are tangent to S. Let $\gamma_{AB} = g|_S(\vec{e}_A, \vec{e}_B)$ denote the first fundamental form of S in $(\mathcal{V}, g)$, which is assumed to be positive definite so that S is spacelike. The standard splitting into tangential and normal directions to S leads to the typical formula relating the covariant derivatives on $(\mathcal{V}, g)$ and (S, γ) [10, 11]:

$$\nabla_{\vec{e}_A}\vec{e}_B = \overline{\Gamma}^C_{AB}\vec{e}_C - \vec{K}_{AB}$$

where $\overline{\Gamma}^C_{AB}$ are the coefficients of the Levi-Civita connection $\overline{\nabla}$ of γ (i.e. $\overline{\nabla}\gamma = 0$), and $\vec{K}_{AB}$ is the extrinsic curvature vector of S, which is orthogonal to S. From here it is immediate to obtain the standard relation

$$\forall \boldsymbol{v} \in T^*(\mathcal{V}), \qquad \Phi^*(\nabla \boldsymbol{v}) = \overline{\nabla}(\Phi^* \boldsymbol{v}) + \boldsymbol{v}(\vec{K}) \tag{1}$$

with Φ^* the pull-back of Φ. Equivalently, (1) can be written as $(\mu, \nu, \ldots = 0, 1, \ldots, D-1)$

$$e^\mu_A e^\nu_B \nabla_\mu v_\nu|_S = \overline{\nabla}_A \overline{v}_B + v_\mu|_S K^\mu_{AB}, \qquad \overline{v}_B \equiv v_\mu|_S\; e^\mu_B\,. \tag{2}$$

For the purposes of this work, the best characterization of trapped surfaces is provided by the causal character of their mean curvature vector $\vec{H}$ [9, 10], which is defined as the trace of the extrinsic curvature vector

$$\vec{H} \equiv \gamma^{AB}\vec{K}_{AB}\,.$$

Its virtue is that it can be clearly generalized to imbedded submanifolds S of any co-dimension, as follows[1]: S is said to be

i. *future trapped* if $\vec{H}$ is timelike and future-pointing all over S. Similarly for *past trapped*.

ii. *nearly future trapped* if $\vec{H}$ is causal and future-pointing all over S, and timelike at least at a point of S, and correspondingly for *nearly past trapped*.

iii. *marginally future trapped* if $\vec{H}$ is null and future-pointing all over S, and non-zero at least at a point of S, and analogously for *marginally past trapped*.

iv. *extremal* or *symmetric* if $\vec{H} = \vec{0}$ all over S.

v. *absolutely non-trapped* if $\vec{H}$ is spacelike all over S.

In codimension two, points (i), (iv) and (v) coincide with the standard nomenclature; point (ii) is a new term coined in [8], while (iii) is more general than the standard concept (e.g. [3, 4]) —still, all standard marginally trapped $(D-2)$-surfaces are included in (iii)—. On the other hand, the above terminology is unusual for the cases $d = D-1$ or $d = 1$.

[1] The zero vector $\vec{0}$ is null; "causal vector" is used for non-spacelike ones.

In the first possibility S is a spacelike hypersurface and thus (i) corresponds to the case where its expansion has a sign everywhere on S, (ii) to the case where it is non-negative or non-positive, (iv) to a maximal S, while (iii) and (v) are impossible. Similarly, S represents a spacelike curve when $d=1$ and therefore the concept of trapping means here that the proper acceleration vector of the curve is timelike along S for (i), null for (iii), and that S is a geodesic for (iv).

We are now ready to obtain the main formula. To that end, let $\vec{\xi}$ be an arbitrary C^1 vector field on $\mathcal{V}$ defined on a neighbourhood of S. Recalling the identity $(\pounds_{\vec{\xi}}\, g)_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$ where $\pounds_{\vec{\xi}}$ denotes the Lie derivative with respect to $\vec{\xi}$, it trivially follows on using (2) that

$$\pounds_{\vec{\xi}}\, g|_S(\vec{e}_A, \vec{e}_B) = \overline{\nabla}_A \overline{\xi}_B + \overline{\nabla}_B \overline{\xi}_A + 2\xi_\mu|_S K^\mu_{AB}$$

and taking the trace here

$$\boxed{\frac{1}{2}\gamma^{AB}\pounds_{\vec{\xi}}\, g|_S(\vec{e}_A, \vec{e}_B) = \overline{\nabla}_C \overline{\xi}^C + (\vec{\xi}\cdot\vec{H})|_S} \tag{3}$$

where $(\,\cdot\,)$ is the g-scalar product.

When $\vec{\xi}$ is a Killing vector the left hand side of (3) vanishes. Integrating the right hand side on S and using Gauss' theorem the divergence term does not contribute *if* S is closed, hence ($\boldsymbol{\eta}_S$ is the canonical volume element d-form associated to γ)

$$\forall \text{ closed } S, \text{ and } \forall \text{ Killing vector } \vec{\xi}: \qquad \int_S (\vec{\xi}\cdot\vec{H})\, \boldsymbol{\eta}_S = 0.$$

Suppose that $\vec{\xi}$ is timelike on S. If S were (nearly, marginally) trapped, $(\vec{\xi}\cdot\vec{H})$ would have a sign where it is not zero on S, which is incompatible with the previous result. Thus

There are no marginally trapped, nearly trapped, nor trapped closed imbedded submanifolds in any stationary region of $(\mathcal{V}, g)$.

As this does not depend on the codimension, in particular one also deduces that closed spacelike curves with a future-pointing (or past) acceleration vector are forbidden in stationary regions, except for the extremal case of closed spacelike geodesics. Similarly, any closed spacelike hypersurface in a stationary spacetime must have an expansion which changes sign, unless the hypersurface is maximal. As a matter of fact, the extreme possibility with $\vec{H}=\vec{0}$ is certainly possible for all d. For instance, asymptotically flat maximal slices always exist in asymptotically flat stationary spacetimes [12], and there are simple examples of extremal S of codimension two in stationary spacetimes [8, 10].

The previous result might seem intuitively obvious at first, it is not straightforward, though, for *there are* trapped surfaces in stationary, and in static, spacetimes. A simple example taken from [4] (p. 776) is provided by the $(D-2)$-surfaces $e^{x^0} - \cosh(x^1) = c_1$, $x^2 = c_2$ with $c_1, c_2 =$const. in flat Minkowski spacetime with Cartesian coordinates $\{x^\mu\}$. Thus, the requirement that the submanifolds be closed is indispensable to get the result.

Suppose now that the Killing vector $\vec{\xi}$ is null and non-zero on the closed S. Then again if S were trapped, or nearly trapped, $(\vec{\xi}\cdot\vec{H})$ would be not identically zero and non-positive (or non-negative) all over S in contradiction with the previous result. However, S can be marginally trapped if $\vec{\xi}$ and $\vec{H}$ are proportional, giving $(\vec{\xi} \cdot \vec{H}) = 0$. Then,

If $\vec{\xi}$ is any Killing vector which is null and non-zero on a closed S, then S cannot be trapped nor nearly trapped, and can be marginally trapped if and only if its mean curvature vector is parallel to $\vec{\xi}$ everywhere.

A direct consequence of this statement is that pp-wave spacetimes (see e.g. [13]) do not admit everywhere expanding (contracting) closed spacelike hypersurfaces, nor closed trapped or nearly trapped submanifolds. And any closed marginally trapped submanifold must be contained in one of the null hypersurfaces orthogonal to the null covariantly constant Killing vector [8].

Note that the proofs of the previous results do not need the Killing property for $\vec{\xi}$: it is enough that just the part of $\pounds_{\vec{\xi}}\, g$ tangent to S is traceless so that the left hand side of (3) vanishes. This allows for many generalizations, among them those using the generalized symmetries introduced recently in [14], see the explanations in [8]. Furthermore, even when the part of $\pounds_{\vec{\xi}}\, g$ tangent to S is not trace-free one can extract useful information from formula (3). Consider the important example of conformal Killing vectors $\pounds_{\vec{\xi}}\, g = 2\phi g$ so that $\frac{1}{2}\gamma^{AB}\pounds_{\vec{\xi}}\, g|_S(\vec{e}_A, \vec{e}_B) = \phi d$. If S is closed, integrating (3) one gets

$$\forall \text{ closed } S, \text{ and } \forall \text{ conformal Killing vector } \vec{\xi}: \qquad \int_S \phi\, \boldsymbol{\eta}_S = \frac{1}{d}\int_S (\vec{\xi} \cdot \vec{H})\, \boldsymbol{\eta}_S.$$

It follows that if $\vec{\xi}$ is timelike and $\phi|_S = (1/D)\nabla_\mu \xi^\mu|_S$ does not vanish, the mean curvature vector of any (nearly, marginally) trapped closed S must be causally oriented in such a way that $\text{sign}(\vec{\xi} \cdot \vec{H})|_S$ and $\text{sign}\, \phi|_S$ coincide where they are non-zero —all timelike homothetic vectors are here included as they have a constant ϕ—. Furthermore, extremal closed submanifolds are forbidden. As an example one can use the timelike conformal Killing vector $\vec{\xi} = a(t)\partial_t$ of any Robertson-Walker spacetime, where $a(t)$ is the scale factor and t the preferred time coordinate [13]. Given that $\pounds_{\vec{\xi}}\, g = 2\dot{a} g$, all possible closed (nearly, marginally) trapped submanifolds must be *past-trapped* if the Robertson-Walker spacetime is *expanding*, and vice versa. Besides, in neither case there can be extremal closed surfaces, nor maximal closed hypersurfaces; and there cannot be closed spacelike geodesics.

Coming back to the general situation, assume now that S is a submanifold (closed or otherwise) orthogonal to the vector field $\vec{\xi}$. Then $\overline{\xi}_C = 0$ and formula (3) simplifies to $\gamma^{AB}\pounds_{\vec{\xi}}\, g|_S(\vec{e}_A, \vec{e}_B) = 2(\vec{\xi} \cdot \vec{H})|_S$. Hence, if $\vec{\xi}$ is a Killing vector (or any of the afore mentioned generalizations), $(\vec{\xi} \cdot \vec{H})|_S = 0$. In particular, if $\vec{\xi}$ is timelike (respectively null), then any S orthogonal to $\vec{\xi}$ must be absolutely non-trapped or extremal (resp. or marginally trapped with $\vec{\xi}$ and $\vec{H}$ parallel), or a mixture thereof. These results have special relevance for the case of *static* regions of the spacetime, where there is a hypersurface-orthoghonal timelike Killing vector [8].

Combining all the previous results the following statements hold:

In stationary regions of $(\mathcal{V}, g)$, any marginally trapped, nearly trapped, or trapped submanifold is necessarily non-closed and non-orthogonal to the timelike Killing vector.

In regions with a null Killing vector $\vec{\xi}$, all trapped or nearly trapped submanifolds must be non-closed and non-orthogonal to $\vec{\xi}$, and any marginally trapped submanifold must have a mean curvature vector parallel (and orthogonal!) to the null Killing vector.

Let us make some comments on why the generalization to arbitrary dimension may be important. Apart from the obvious reasons (many results can be derived in a simple and direct manner for closed spacelike curves, or closed spacelike slices, etcetera), its relevance arises because it may serve to generalize the singularity theorems [1, 2, 3, 4] to arbitrary dimension and to many new situations. This application can be grasped from the following situation. Consider a 4-dimensional spacetime with whole cylindrical symmetry [13, 15], that is to say, such that there exist local coordinates in which the line-element reads

$$ds^2 = -A^2dt^2 + B^2d\rho^2 + C^2d\varphi^2 + E^2dz^2, \tag{4}$$

where $\partial_\varphi, \partial_z$ are spacelike commuting Killing vectors whence all the functions depend only on t and ρ, and the axis is located at $\rho \to 0$. The coordinate φ is closed with standard periodicity 2π. The geometrically preferred 2-surfaces are the cylinders given by constant values of t and ρ, and it is very easy to check whether they are trapped surfaces or not: it is enough to compute the causal character of the gradient of the function CE —see below for a justification of this. However, these cylinders are *not* closed in general, so that whether or not they are trapped has no *direct* relevance for the development of singularities in the sense of geodesic incompleteness.

Nevertheless, the spacelike curves defined by constant values of t, ρ and z are certainly *closed*, and one wonders if the trapping of these curves can lead to similar conclusions, regarding incompleteness of causal geodesics, as the trapping of 2-surfaces do in the classical singularity theorems. However, the theory of points focal to any given spacelike d-submanifold along causal geodesics is much more involved if $D - d \neq 1, 2$, because the variation vectors orthogonal to the geodesic are not, *all of them*, tangent to the submanifold (as opposite to the cases of codimension 1 for timelike geodesics and codimension 2 for null ones). Thus, the typical results on variation of arc-length [3, 16, 4] and maximal geodesics are unclear, and the focusing effect requires additional restrictions on the curvature (not only the null convergence or weak energy conditions). In spite of all this, there is a kind of underlying common property for submanifolds of all codimensions, as has been shown here with the unifying definition for them, so that perhaps a conjecture along the lines of: "If appropriate restrictions on the curvature hold and there exists a closed future (past) trapped submanifold S, then either $E^+(S)$ ($E^-(S)$) is compact, or the spacetime is null geodesically incomplete to the future (past), or both" may be true. If this held, then by assuming also the strong causality condition one could prove that either $E^+(S) \cap S$ ($E^-(S) \cap S$) is a trapped set, or $(\mathcal{V}, g)$ is geodesically incomplete, (see [2, 3, 4] for the definition of $E^\pm(S)$, the concept of trapped set, and the other conditions mentioned here). The previous conjecture is well known to be true for the case of codimension two (and one, trivially) by assuming simple "energy conditions" [3, 16, 4] on the Ricci part of the curvature tensor. If the conjecture became concreted, and could thus be proved, then the Hawking-Penrose singularity theorem [2, 3, 4] could be generalized to the case of having a closed trapped submanifold of any codimension in arbitrary dimension very easily.

This is no place to discuss the conjecture in general, but in order to support its plausibility, let us come back to the cylindrically symmetric spacetimes. First of all, it is a matter of simple calculation to know when the closed curves mentioned before are trapped or not, because minus their proper acceleration vector (that is, the mean curvature vector) is easily seen to be $\vec{H} = C^{-2}(A^{-2}C_{,t}\,\partial_t - B^{-2}C_{,\rho}\,\partial_\rho)$. Therefore, $(\vec{H}\cdot\vec{H}) = C^{-4}(dC\cdot dC)$, so that the causal character of the gradient of *only* the function C is what matters here. A straightforward consequence is that small circles close to the axis are always absolutely non-trapped, because if the axis is regular the functions behave as $C^2 \sim f(t)\rho^2 + O(4)$, $B^2 \sim f(t) + O(2)$ and $A^2 \sim h(t) + O(2)$ when $\rho \to 0$. More importantly, we are now ready to check if the conjecture is valid in some simple examples. To that end, take the following particular cylindrically symmetric spacetimes, described in their form (4) by [17, 4]:

$$A = B = T^2(t)\cosh(3a\rho), \;\; C = T^2(t)\frac{\sinh(3a\rho)}{3a}\cosh^{-1/3}(3a\rho), \;\; E = T^{-1}(t)\cosh^{-1/3}(3a\rho)$$

where a is a constant and the function $T(t)$ can be either $\cosh(at)$ or $\sinh(at)$. In both cases, the metric is a solution of Einstein's field equations for a comoving perfect-fluid energy-momentum tensor with realistic equation of state $p = \varrho/3$, being the pressure $p = 5a^2T^{-4}(t)\cosh^{-4}(3a\rho)$. These spacetimes are globally hyperbolic and the dominant energy condition is obviously satisfied. An elementary calculation of dC and its causal character permits to show that the φ-circles are closed absolutely non-trapped curves in the case $T(t) = \cosh(at)$, while there always are closed trapped φ-circles for some values of t, ρ in the other case $T(t) = \sinh(at)$. This is in agreement with the conjecture for, as is known [17, 4], the first case defines a geodesically complete spacetime while the second one is past geodesically incomplete—it has a big-bang singularity at $t = 0$—.

Let us finally concentrate on the historically relevant case of codimension two. It is well-known that for the case of spherically symmetric spacetimes, i.e. such that the line-element can be expressed as

$$ds^2 = g_{ab}(x^c)dx^a dx^b + R^2(x^c)d\Omega^2_{D-2}\,,$$

where $\{x^a\}$ $(a, b, \ldots = 0, 1)$ are coordinates on the Lorentzian 2-dimensional metric g_{ab} and $d\Omega^2_{D-2}$ is the round metric on the $(D-2)$-sphere, the causal character of the gradient of the function R fully characterizes the trapping or not of the spheres, and strongly affects that of any other closed submanifold of codimension 2 (e.g. [9]). The question arises of how to generalize this simple criterion to arbitrary spacetimes. This was studied in [9], where the following simple way to check the trapping of surfaces was presented. Take any given family of $(D-2)$-dimensional spacelike surfaces S_{X^a} described by $\{x^a = X^a\}$ where X^a are constants and $\{x^\alpha\}$ are local coordinates in $\mathcal{V}$. The line-element can be locally written as

$$ds^2 = g_{ab}dx^a dx^b + 2g_{aA}dx^a dx^A + g_{AB}dx^A dx^B \tag{5}$$

where $\det g_{AB} > 0$ and all the g-components can depend on *all* coordinates: $g_{\mu\nu}(x^\alpha)$. Define

$$e^U \equiv +\sqrt{\det g_{AB}}\,, \qquad\qquad \boldsymbol{g}_a \equiv g_{aA}dx^A\,.$$

Then, the mean curvature one-form of S_{X^a} reads ($f_{,a} = \partial_{x^a} f$)

$$\boxed{H_\mu = \delta^a_\mu \left(U_{,a} - \mathrm{div}\vec{g}_a\right)|_{S_{X^a}}} \tag{6}$$

where div is the divergence operating on vectors at each S_{X^a}, so that its Lorentz length is

$$g^{\mu\nu}|_{S_{X^a}} H_\mu H_\nu = g^{bc}|_{S_{X^a}} H_b H_c = g^{bc}\left(U_{,b} - \mathrm{div}\vec{g}_b\right)\left(U_{,c} - \mathrm{div}\vec{g}_c\right)\Big|_{S_{X^a}} . \tag{7}$$

(Notice that g^{bc} is not necessarily the inverse matrix of g_{bc}!). Contracting (6) with a vector field $\vec{\xi}$ one obtains an expression for $(\vec{\xi} \cdot \vec{H})$ which can be related, after some calculations, to (3) (when $d = D-2$). Note also that one only needs to compute the norm of $\boldsymbol{H}$ *as if it were a one-form in the 2-dimensional metric* g^{ab}. Thus, (7) provides the sought criterion. Many applications arise, for a selection of them the reader is referred to [9].

Acknowledgements

This contribution is largely based on a collaboration with Marc Mars, to whom I am also grateful for many comments. Financial support under grants BFM2000-0018 of the Spanish CICyT and no. 9/UPV 00172.310-14456/2002 of the University of the Basque Country is acknowledged.

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