On local existence of dynamical horizons

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Abstract

Given a foliation $\mathcal{N}(t)$ of spacetime by spacelike hypersurfaces and a marginally trapped surface $\mathcal{S}(t_0)$ on the initial slice $\mathcal{N}(t_0)$, we discuss the problem of local existence of a "dynamical horizon" through $\mathcal{S}(t_0)$ whose marginally trapped leaves lie on the $\mathcal{N}(t)$.

1 How does a marginally trapped surface evolve ?

The crucial objects in the studies of gravitational collapse by Hawking and Penrose are future- or past trapped 2-surfaces (TS). They are defined by sign conditions on the expansions θ^+ and θ^- of the divergencies of null geodesic congruences emanating orthogonally from the surface. A marginally trapped surfaces can be defined either as the boundary of all TS lying in a given 3-dimesional hypersurface $\mathcal{N}(t_0)$, or as special TS on which (at least) one of the null geodesic expansions vanishes. As shown by Kriele and Hayward [1], a boundary of the set of TS has indeed vanishing null geodesic expansion provided it is a differentiable surface, which we assume henceforth.

Consider a spacetime $\mathcal{M}$ and a spacelike slice $\mathcal{N}(t_0)$. If the latter contains a MTS, we expect that the same is true for all slices $\mathcal{N}(t)$ at slightly later times $t > t_0$, in which case the MTS would locally span a 3-surface $\mathcal{D}$. Assuming the existence of such surfaces, Hayward [2] (who called them "trapping horizons") studied some of their properties and a more detailed analysis was recently carried out by Ashtekar and Krishnan [3] who call essentially the same objects "dynamical horizons"; we adopt this terminology (using the shorthand DH). In this talk I report on mathematical studies on the existence problem for DH obtained in joint work with Lars Andersson and Marc Mars.

An important first issue to note in this enterprise is that there is in general no unique DH through a given MTS. Instead, for every prescribed foliation $\mathcal{N}(t)$ of some portion of spacetime which has a MTS $\mathcal{S}(t_0)$ on the initial slice $\mathcal{N}(t_0)$, one expects to find a DH whose marginally trapped leaves $\mathcal{S}(t)$ lie on $\mathcal{N}(t)$.

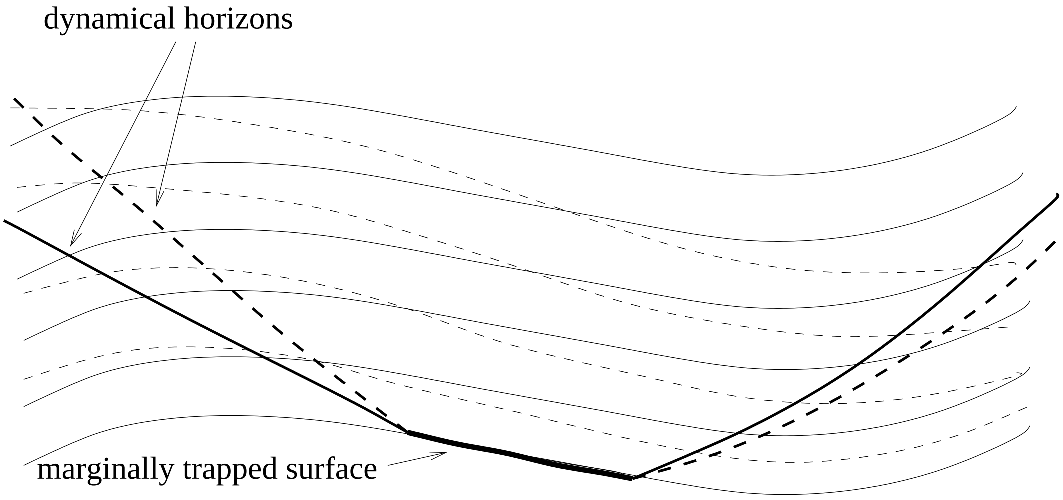

Figure 1: Dynamical horizons for different foliations

Changing the foliation $\mathcal{N}(t)$ for $t > 0$ will lead to a different DH in general, as shown in figure 1. (Otherwise the DH would be foliated by at least two different families of MTS). This also means that the DH arising in this way will in general be different from the boundary of the set in spacetime which contains TS.

A possible route for proving the existence of DH is illustrated in Fig. 2. Consider a null geodesic congruence emanating orthogonally from the MTS $\mathcal{S}(t_0)$. It will cut out surfaces $\widetilde{\mathcal{S}}(t)$ on $\mathcal{N}(t)$ for $t > t_0$ which, by virtue of Raychaudhuri's equation, will be trapped provided the energy conditions hold. If, moreover, the $\mathcal{N}(t)$ are asymptotocally flat, they will contain "almost round" untrapped surfaces at least in the asymptotic region. Argueing heuristically, if "everything" is sufficiently smooth there will be MTS somewhere between the surface $\widetilde{\mathcal{S}}(t)$ and the asymptotic region, and we also see that the resulting DH is either spacelike or null. It should be possible to make this argument rigorous e.g. by considering the equation of a MTS as a graph over $\widetilde{\mathcal{S}}(t)$ in $\mathcal{N}(t)$ (c.f. [1]).

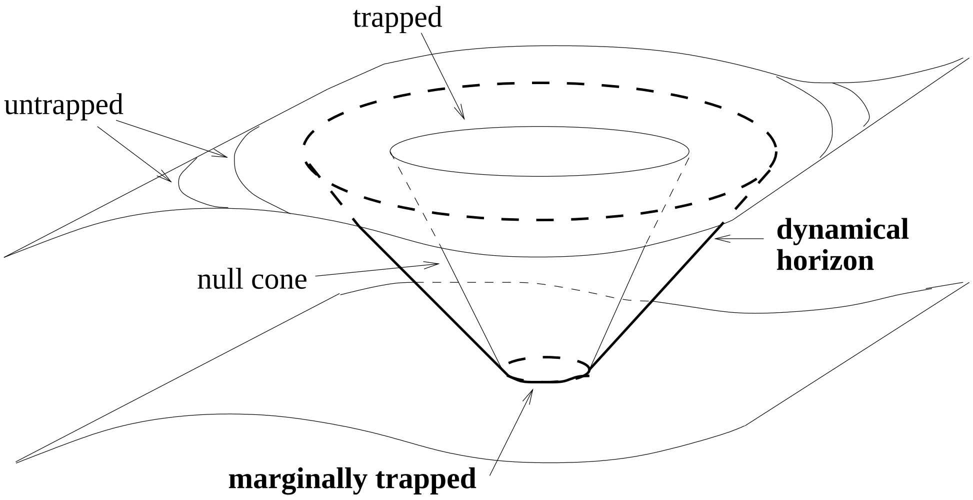

Figure 2: On existence of dynamical horizons

An alternative formulation is to assume that the spacetime foliation $\mathcal{N}(t)$ is given in terms of the "lapse" along the normal direction, and the problem is to find an equation for the "shift" tangential to each leaf which determines the location of the DH. In this talk I discuss the "infinitesimal" version of this problem which only involves simple geometric quantities on the initial MTS $\mathcal{S}(t_0)$. On the other hand, one obtains in this way only *necessary* rather than *sufficient* condition for the existence of the DH through $\mathcal{S}(t_0)$. Before I describe this procedure, I consider in sections 2 and 3 the more special problem of finding conditions under which a given initial data surface can be foliated by MS or by MTS.

2 When can a 3-manifold be foliated by minimal surfaces ?

Let $(\mathcal{N}, h_{ij})$ be a Riemannian manifold and let $\mathcal{S}(r_0)$ be a minimal surface (MS) on $\mathcal{N}$. In general, MS are isolated but in special cases as for the cylinder in Fig. 3 there obviously exists a foliation by MS. We now want to find general conditions, preferably to be imposed on $\mathcal{S}(r_0)$ only, which are necessary and possibly also sufficient in order for $\mathcal{N}$ to be foliated by minimal surfaces near $\mathcal{S}(r_0)$.

Let D_i be the covariant derivative on $\mathcal{N}$ and let ${}^3R$ denote the scalar curvature of $(\mathcal{N}, h_{ij})$. We first introduce an arbitrary foliation $\mathcal{S}(r)$ (i.e. the level surfaces are $r = const.$), and denote its unit normal vector by

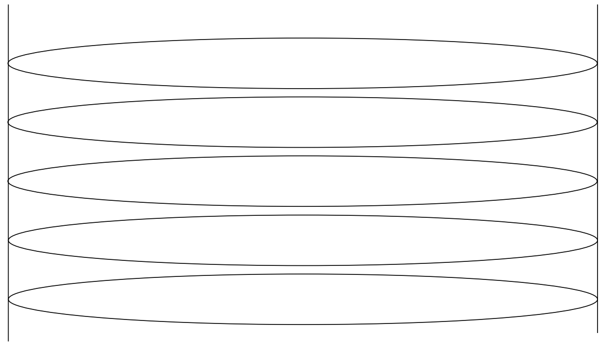

Figure 3: A manifold foliated by minimal surfaces

m^i and the "lapse" by M, i.e. $Mm^iD_ir = 1$. For each level surface $\mathcal{S}(r)$ let $p_{ij} = h_i^k h_j^l D_k n_l$ be the second fundamental form, $p = p_{ij}q^{ij}$ the mean curvature, $u_{ij} = p_{ij} - 1/2pq_{ij}$ the trace-free part of p_{ij}, and ${}^2R$ the scalar curvature. The following formula can be obtained by making use of the Gauss-Codazzi equation and by specifying to a minimal surface $p = 0$,

$$\frac{dp}{dr}|_{p=0} = -{}^2\triangle M + \frac{1}{2}({}^2R - {}^3R - u_{ij}u^{ij})M. \tag{1}$$

Requiring now that $dp/dr = 0$ yields the *linear homogeneous elliptic* equation

$${}^2\triangle M + B_M.M = 0 \tag{2}$$

for the lapse M, where the definition of B_M can be read off from (1). Dividing (2) by M and integrating by parts gives

$$q^{ij}D_i(M^{-1}D_jM) = -M^{-2}q^{ij}(D_iM)(D_jM) + B_M. \tag{3}$$

In general, (2) will have no solutions on the compact domain $\mathcal{S}$. The existence of solutions is clearly a necessary but in general not a sufficient criterium for the existence of a local foliation by MS.

Equation (3) can be integrated over $\mathcal{S}$, using the Gauss-Bonnet theorem for the integral of ${}^2R$ which gives $8\pi(1-g)$ where g is the genus of $\mathcal{S}$. Requiring the "energy condition" ${}^3R \geq 0$ shows that this particular "dynamical horizon" must have spherical or toroidal topology. We recall that, more generally, (2) and (3) were used by Hawking [4] and by Cai and Galloway [5] to prove that *outermost* minimal surfaces must also have that topology. The argument in essence consists of the observation that outermost MS are "stable" in the sense that the lowest eigenvalue λ of the "stability operator" ${}^2\triangle + B_M$

must be non-negative, and again of dividing (1) by M and integrating over $\mathcal{S}$. The condition (2) for a foliation by MS to exist is equivalent to $\lambda = 0$. Though this foliation would be called "stable" in mathematical terminology, the given MS could nevertheless "decay" to other leaves of the foliation.

3 When can an initial data set be foliated by marginally trapped surfaces ?

We now generalize the above reasoning from minimal to general marginally trapped surfaces, still staying within a given 3-surface. We consider an initial data set $(\mathcal{N}, h_{ij}, k_{ij})$ whose second fundamental form k_{ij} is decomposed as follows with respect to an arbitrary foliation with unit normal vector m^i,

$$k_{ij} = z m_i m_j + m_i s_j + m_j s_i + q_i^k q_j^l x_{kl} + \frac{1}{2} q q_{ij}, \tag{4}$$

where $z = k_{ij} m^i m^j$, $s_i = q_i^j k_{jl} m^l$, $q = k_{ij} q^{ij}$ and $x_{ij} = q_i^l q_j^n k_{ln} - (1/2) q q_{ij}$. (We have here adopted the notation of [6] except for quantities already defined above).

In terms of these quantities we can write the expansions as $\theta^{\pm} = p \pm q$. We obtain, from equ. (10) of [6] and by setting $\theta^+ = 0$ on the r.h. side, the following generalization of (1):

$$\begin{aligned} \frac{d\theta^+}{dr}|_{\theta^+=0} &= -{}^2\triangle M + 2 q^{ij} s_j D_i M + \\ &+ \frac{1}{2}[{}^2R - \Pi - (x_{ij} - u_{ij})(x^{ij} - u^{ij}) - s_i s^i + q^{ij} D_i s_j] M \end{aligned} \tag{5}$$

where $\Pi = {}^3R - k_{ij}k^{ij} + k^2 - 2 m^j D_i (k_j^i - k\delta_j^i)$.

The necessary condition $d\theta^+/dr = 0$ for the existence of a local foliation by MTS can now be written as

$${}^2\triangle M + A^i_{MT} D_i M + B_{MT} M = 0 \tag{6}$$

where the definitions of A^i_{MT} and B_{MT} can be read off from (5). Alternatively, we can write (6) in the same form as (3) which yields

$$\begin{aligned} & q^{ij} D_i (D_j \ln M - s_j) = \\ & \quad - q^{ij} (D_i \ln M - s_i)(D_j \ln M - s_j) + \frac{1}{2}[{}^2R - \Pi - (x_{ij} - u_{ij})(x^{ij} - u^{ij})]. \end{aligned} \tag{7}$$

Like (2), (6) is again *linear elliptic* and *homogeneous* and will have no solutions in general.

Integrating (7) and assuming the "energy condition" $\Pi \geq 0$ to hold again shows that dynamical horizons must have spherical or toroidal topology (as observed by [3]), and the same is known to be true for *outermost* MTS [4, 5].

4 The general case

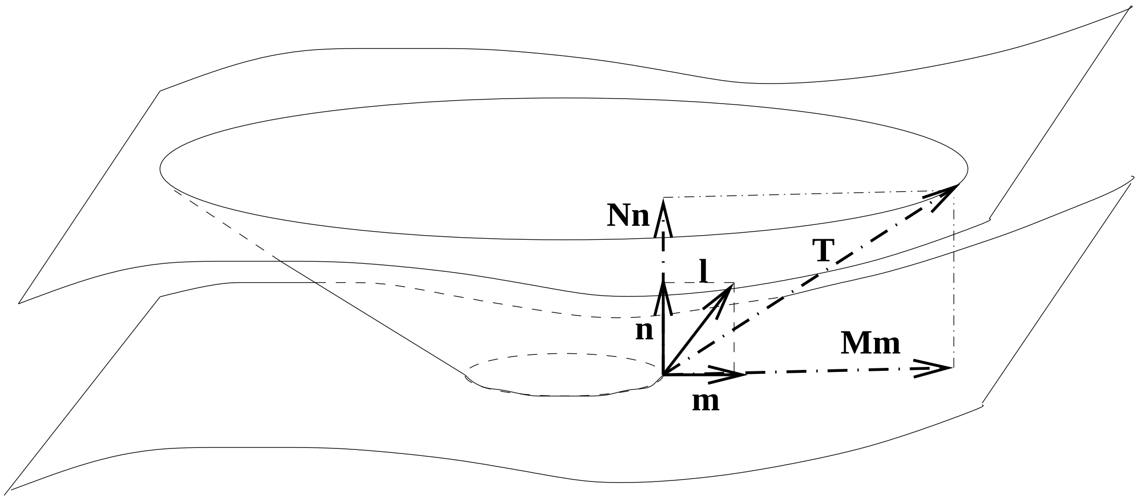

Figure 4: A dynamical horizon

Finally we examine how this picture changes when we allow the MTS to "evolve in spacetime". To do so we rewrite some of the above notation and equations introduced above on 3-manifolds to spacetime language (c.f. Fig. 4). We work on a Lorentzian manifold $(\mathcal{M}, g_{\mu\nu})$ (signature $(-,+,+,+)$) and denote by ∇_μ the covariant derivative of $g_{\mu\nu}$. Let $\mathcal{N}(t)$ be a foliation of some part of $\mathcal{M}$ by spacelike hypersurfaces $t = const.$, with n^μ and N denoting the unit normal and the "lapse" function, i.e. $Nn^\mu \nabla_\mu t = 1$. Further, let $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$, $k_{\mu\nu} = h^\sigma_\mu h^\tau_\nu \nabla_\sigma n_\tau$ and $k = h^{\mu\nu} k_{\mu\nu}$ denote the induced metric, the second fundamental form and the mean curvature of $\mathcal{N}(t)$, respectively, and the covariant derivative on the $\mathcal{N}(t)$ is called D_μ. Let $\mathcal{S}(t,r)$ be a family of compact 2-surfaces without boundary lying for $t = const.$ in $\mathcal{N}(t)$, and labeled by $r = const.$, for r in some small interval. The outward pointing unit normal to $\mathcal{S}(t,r)$ tangent to $\mathcal{N}(t)$ is called m^μ, the corresponding lapse M is given by $Mm^\mu \nabla_\mu r = 1$, and the induced metric on each $\mathcal{S}(t,r)$ is denoted by $q_{\mu\nu} = h_{\mu\nu} - m_\mu m_\nu$.

Let $\mathcal{D}$ be the spacelike 3-surface determined by $r = const.$. Then the vector field $T^\mu = Nn^\mu + Mm^\mu$ is tangent to $\mathcal{D}$ and satisfies $T^\mu \nabla_\mu t = 1$, and the unit tangent vector is called t^μ. Further, let $U^\mu = Mn^\mu + Nm^\mu$ be a normal to $\mathcal{D}$, and let u^μ be the unit normal. We also introduce the null vector $l^\mu = m^\mu + n^\nu$, the energy-momentum tensor $T_{\mu\nu}$ and the null second fundamental form $\theta_{\mu\nu} = q_\mu^\tau q_\nu^\sigma \nabla_\sigma l_\tau$ whose trace on $\mathcal{S}(t,r)$ also yields the expansion $\theta^+ = q^{\mu\nu}\theta_{\mu\nu}$. Finally, we define the vector $W_\mu = q_\mu^\tau t^\sigma \nabla_\sigma l_\tau$ which is tangent to the surfaces $\mathcal{S}(t,r)$.

We can now either rewrite either (7) on spacetime or equ. (24) of Cai and Galloway [5] (setting $d\theta/dt = 0$) to obtain

$$q^{\mu\nu}\nabla_\nu W_\mu = -q^{\mu\nu}W_\mu W_\nu + \frac{1}{2}({}^2R - 2T_{\mu\nu}u^\mu l^\nu - \theta_{\mu\nu}\theta^{\mu\nu}). \tag{8}$$

We are now going to use this equation to determine the location of the DH for a spacetime with a prescribed foliation. More precisely, we prescribe the "lapse" N of the foliation and determine the "shift" M such that $T^\mu = Nn^\mu + Mm^\mu$ is a tangent to the DH and (hence) $U^\mu = Nm^\mu + Mn^\mu$ is normal. A computation shows that, in terms of these quantities,

$$W_\mu = \frac{1}{\sqrt{M^2 - N^2}}[-q_\mu^\tau \nabla_\tau (M+N) + (M-N)q_\mu^\tau k_{\tau\sigma} m^\sigma]. \tag{9}$$

Inserting this expression in (8) we can rewrite the latter and obtain an equation for M of the form

$${}^2\triangle M + A^i D_i M + B.M = C \tag{10}$$

where A^i, B and C depend on the given quantities N, $D_i N$, ${}^2\triangle N$ on m^i and on the second fundamental forms k_{ij} and p_{ij} but also on M and $D_i M$ in a rather complicated way; in particular, the A^i depend on $D^i M$. For $N = 0$ the equation reduces to (6) and setting also $k_{ij} = 0$ we obtain (2). However, in contrast to (2) and (6), (10) is *semilinear* and *inhomogeneous*.

In Sect.1 we have put forward arguments in favour of the existence of DH with respect to given foliations of spacetime. Accordingly, we expect that solutions to (10) exist in general, and this expectation is, on a heuristic level, supported by the inhomogeneity of the equation. Suppose for the moment that the equation were linear, i.e. that A^i, B and C are independent of M and $D^i M$. Then the Fredholm alternative would guarantee that there either exist solutions to the homogeneous equation ((10) with $C = 0$) or to (10) as it stands. For certain classes of nonlinear equations similar arguments show that a trivial kernel of the elliptic operator guarantees existence of solutions. Putting now $C = 0$ in (10) gives an equation very similar (but not identical) to (6), and recall that solutions to the latter are necessary for the existence of a foliation of the given initial surface by MTS. This reasoning indicates that (10) will indeed have solutions unless $C = 0$ in which case the the given foliation would be degenerate and "resemble" a dynamical horizon near the given MTS $\mathcal{S}(r_0)$. We intend to provide a rigorous argument for existence of solutions along these lines elsewhere.

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