GRAVITATIONAL WAVES AND EQUATIONS OF MOTION: POST-NEWTONIAN METHODS

Clifford M. Will
Mathematics of Gravitation II, Warsaw, Sept. 1-9, 2003

- INTRODUCTION
- THE DIRE PROGRAM
  - Direct Integration of the Relaxed Einstein Equations
  - 3.5PN Equations of Motion: Structure Dependence?
  - Spin and Radiation Reaction
- THE END-GAME OF BINARY INSPIRAL
  - Post-Newtonian Diagnostics for Numerical Relativity
  - Quasi-Equilibrium Configurations: Circular or Eccentric?

THE RELAXED EINSTEIN EQUATIONS

Field Definition:

$$h^{\alpha\beta} = \eta^{\alpha\beta} - \sqrt{-g} g^{\alpha\beta}$$

Harmonic Coordinate Condition:

$$\partial_\beta h^{\alpha\beta} = 0$$

"Relaxed" Field Equations:

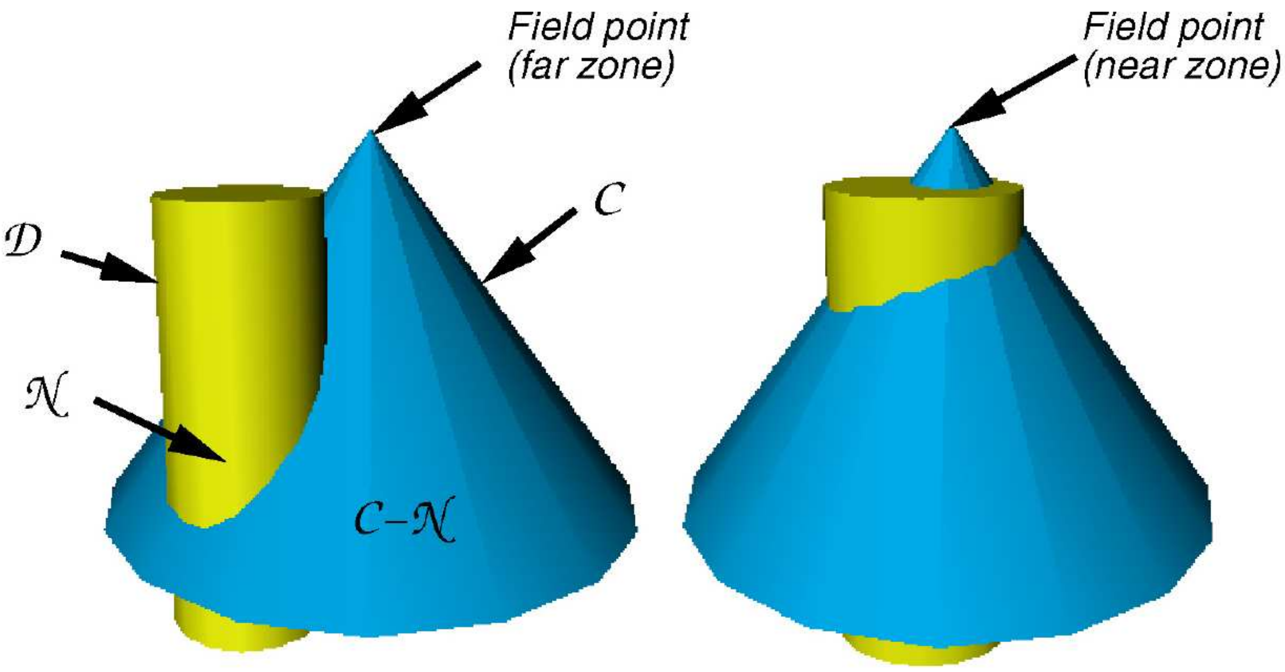
$$\Box h^{\alpha\beta} = -16\pi(-g)T^{\alpha\beta} - \Lambda^{\alpha\beta}(h) \equiv -16\pi\tau^{\alpha\beta}$$

$$\partial_\beta \tau^{\alpha\beta} = 0$$

Formal Solution:

$$\boxed{h^{\alpha\beta} = 4\int_{\mathcal{C}} \frac{\tau^{\alpha\beta}(t', \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \delta(t' - t + |\mathbf{x} - \mathbf{x}'|) d^4x'}$$

Divide into Inner ($\mathcal{N} = \{|\mathbf{x}'| < \mathcal{R}\}$) and Outer ($\mathcal{C} - \mathcal{N} = \{|\mathbf{x}'| > \mathcal{R}\}$) integrals, where $\mathcal{R} \sim \lambda$.



Field point
(far zone)
Field point
(near zone)
$\mathcal{D}$
$\mathcal{C}$
$\mathcal{N}$
$\mathcal{C}-\mathcal{N}$

FAR ZONE SOLUTION: NEAR ZONE INTEGRAL ($r > \mathcal{R}$)

$$\boxed{\begin{aligned} h^{\alpha\beta}_{\mathcal{N}}(t,\mathbf{x}) &= 4\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{\partial^m}{\partial x^{k_1}\dots\partial x^{k_m}}\left(\frac{1}{r}M^{\alpha\beta k_1\dots k_m}\right) \\ M^{\alpha\beta k_1 k_2\dots k_m} &\equiv \int_{\mathcal{M}}\tau^{\alpha\beta}(t-r,x')x^{k_1}\dots x^{k_m}d^3x' \end{aligned}}$$

Radiative Fields: Epstein-Wagoner Decomposition

Harmonic Identities ($\tau^{\alpha\beta}{}_{,\beta}=0$):

$$\tau^{ij} = \frac{1}{2}(\tau^{00}x^i x^j)_{,00} + 2(\tau^{k(i}x^{j)})_{,k} - \frac{1}{2}(\tau^{kl}x^i x^j)_{,kl}$$

Multipole Expansion of Radiative Field:

$$h^{ij}_{\mathcal{N}} = \frac{2}{r}\frac{d^2}{dt^2}\sum_{m=0}^{\infty}\hat{N}_{k_1}\dots\hat{N}_{k_m}I^{ijk_1\dots k_m}_{EW}(t-r)$$

Epstein-Wagoner Moments:

$$\begin{aligned} I^{ij}_{EW} &= \int_{\mathcal{M}}\tau^{00}x^i x^j d^3x + I^{ij}_{SURF} \\ I^{ijk}_{EW} &= \int_{\mathcal{M}}(2\tau^{0(i}x^{j)}x^k - \tau^{0k}x^i x^j)d^3x + I^{ijk}_{SURF} \\ I^{ijk_1\dots k_m}_{EW} &= \frac{2}{m!}\frac{d^{m-2}}{dt^{m-2}}\int_{\mathcal{M}}\tau^{ij}x^{k_1}\dots x^{k_m}d^3x \end{aligned}$$

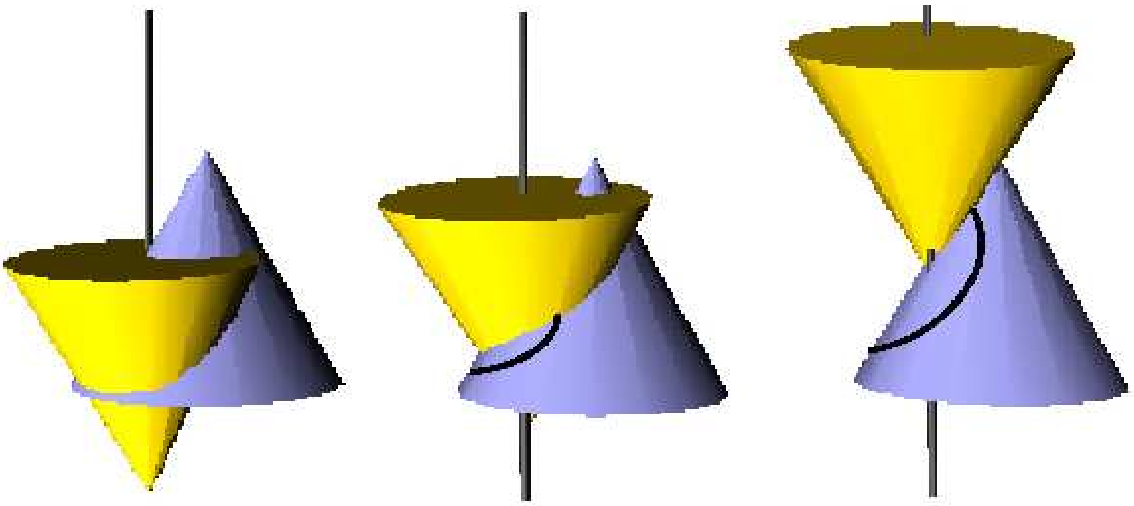
FAR ZONE SOLUTION: FAR ZONE INTEGRAL

Change Variables:

$$u' \equiv t' - r' \qquad \text{or} \qquad t - u' = r' + |\mathbf{x} - \mathbf{x}'|$$

$$r'(u', \Omega') = \frac{r^2 - (t-u')^2}{2(\mathbf{n}' \cdot \mathbf{x} - t + u')}$$

$$\boxed{h^{\alpha\beta}_{\mathcal{C}-\mathcal{N}}(t,x) = 4 \int_{-\infty}^{u} du' \int \frac{\tau^{\alpha\beta}(u'+r', \mathbf{x}')}{t - u' - \mathbf{n}' \cdot \mathbf{x}} [r'(u', \Omega')]^2 d^2\Omega'}$$



$u' < u - 2\mathcal{R}$ $\qquad$ $u - 2\mathcal{R} < u' < u$ $\qquad$ $u' = u$

NEAR ZONE SOLUTION: NEAR ZONE INTEGRAL ($r < \mathcal{R}$)

$$\boxed{h^{\alpha\beta}_{\mathcal{N}}(t,\mathbf{x}) = 4\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{\partial^m}{\partial t^m}\int_{\mathcal{M}}\tau^{\alpha\beta}(t,x')|\mathbf{x}-\mathbf{x}'|^{m-1}d^3x'}$$

Example to 3.5 PN order

$$\begin{aligned}
h^{00}_{\mathcal{N}} &= 4\int_{\mathcal{M}}\frac{\tau^{00}(t,x')}{|\mathbf{x}-\mathbf{x}'|}d^3x' - 4\frac{\partial}{\partial t}\int_{\mathcal{M}}\tau^{00}(t,x')d^3x' \\
&\quad +2\frac{\partial^2}{\partial t^2}\int_{\mathcal{M}}\tau^{00}(t,x')|\mathbf{x}-\mathbf{x}'|d^3x' \\
&\quad -\frac{2}{3}\frac{\partial^3}{\partial t^3}\int_{\mathcal{M}}\tau^{00}(t,x')|\mathbf{x}-\mathbf{x}'|^2d^3x' \\
&\quad +\frac{1}{6}\frac{\partial^4}{\partial t^4}\int_{\mathcal{M}}\tau^{00}(t,x')|\mathbf{x}-\mathbf{x}'|^3d^3x' \\
&\quad -\frac{1}{30}\frac{\partial^5}{\partial t^5}\int_{\mathcal{M}}\tau^{00}(t,x')|\mathbf{x}-\mathbf{x}'|^4d^3x' \\
&\quad +\frac{1}{180}\frac{\partial^6}{\partial t^6}\int_{\mathcal{M}}\tau^{00}(t,x')|\mathbf{x}-\mathbf{x}'|^5d^3x' \\
&\quad -\frac{1}{1260}\frac{\partial^7}{\partial t^7}\int_{\mathcal{M}}\tau^{00}(t,x')|\mathbf{x}-\mathbf{x}'|^6d^3x' + O(\epsilon^5)
\end{aligned}$$

CANCELLATION OF BOUNDARY-DEPENDENT TERMS

We can decompose $\tau^{\alpha\beta}$ into terms of the form

$$\tau^{\alpha\beta} \sim \frac{\hat{n}^{<L>}}{r^B} f_{B,L}(u)$$

For each (B, L):

$$\int_{\mathcal{M}}^{\mathcal{R}} \frac{\tau^{\alpha\beta}(t - |\mathbf{x} - \mathbf{x}'|, x')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \\ \rightarrow \left(\frac{2}{r}\right)^{B-2} \hat{n}^{<L>} \sum_{q=0}^{\infty} r^q \frac{d^q f_{B,L}(u)}{du^q} \mathcal{D}_{\mathcal{N}}(q, B, L, \mathcal{R}/r)$$

Using Maple, for each (B, L):

$$\int_{\mathcal{C}-\mathcal{N}} \frac{\tau^{\alpha\beta}(t - |\mathbf{x} - \mathbf{x}'|, x')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \\ \rightarrow \left(\frac{2}{r}\right)^{B-2} \hat{n}^{<L>} \sum_{q=0}^{\infty} r^q \frac{d^q f_{B,L}(u)}{du^q} \mathcal{D}_{\mathcal{C}-\mathcal{N}}(q, B, L, \mathcal{R}/r)$$

For all $\mathcal{R}$-dependent terms,

$$\boxed{\mathcal{D}_{\mathcal{N}}(q, B, L, \mathcal{R}/r) + \mathcal{D}_{\mathcal{C}-\mathcal{N}}(q, B, L, \mathcal{R}/r) = 0}$$

Cancellation valid for near-zone and far-zone field points, for power of $\mathcal{R}$ and for $\ln \mathcal{R}$.

MATTER MODEL AND EFFECTS OF FINITE SIZE

$$\boxed{\begin{array}{l} T^{\mu\nu} \to \rho u^\mu u^\nu \to \rho^* u^\mu u^\nu / u^0 \sqrt{-g} \\ \rho^* \dfrac{d^2 x^j}{dt^2} + \rho^* \Gamma^j_{\mu\nu} v^\mu v^\nu - \rho^* \Gamma^0_{\mu\nu} v^j v^\mu v^\nu = 0 \end{array}}$$

Finite Size Effects

- Proportional to s^N – "tidal effects" **IGNORE**
- Proportional to s^{-N} – "self-energy" effects **IGNORE**
  - At 1PN order, they renormalized the "mass", i.e. $m \to m + E_{self}$
  - No Nordtvedt effect in GR
  - Evidence that it works at 2PN order (Börner & Rudolph, Grishchuk & Kopeikin)
- Independent of s – "delta-function" terms **KEEP**
- At 3PN order, new terms proportional to s^0, but dependent on internal structure:

$$\boxed{\alpha_4 = \frac{1}{m_A^3} \int_A \rho^* U' X' r^2 d^3 x \to -\frac{38}{105}}$$

STRUCTURE-DEPENDENT TERMS

Two-body potentials: s^0 terms?

$$\begin{aligned} A &= (1/m_1)\int_1 \rho v^2 \nabla^i U U^2 d^3x \\ &\sim v_1^2 \int_1 \rho \nabla^i U_1 \left(U_1^2 + 2U_1U_2 + U_2^2\right) \\ &\sim v_1^2 \int_1 \rho \frac{s^i}{s^3}\left(\frac{1}{s} s^j s^k s^l \nabla^{jkl} U_2 + s^j s^k (\nabla^{jk} U_2) U_2\right) \\ &\sim 0 \end{aligned}$$

Three-body potentials: s^0 terms?

$$\begin{aligned} B &= (1/m_1)\int_1 \rho v^j v^k \nabla^i U P(\nabla^j U \nabla^k U) \\ &\sim (1/m_1) v_1^j v_1^k \int_1 \rho \nabla^i U_1 P(\nabla^j U_1 \nabla^k U_2) \end{aligned}$$

But: $P(\nabla^j U_1 \nabla^k U_2) \sim \rho\rho \nabla^j \nabla^k \ln(r_{12} + r_{1'2} + |s|)$
$\sim \rho\rho \nabla^j |s| \nabla^k (1/r_{12}) \sim \nabla^j X_1 \nabla^k U_2$

$$\begin{aligned} B &\sim (1/m_1) v_1^j v_1^k \int_1 \rho \nabla^i U_1 \nabla^j X_1 s^l s^m \nabla^{klm} U_2 d^3x \\ &\sim m_1^2 m_2 v_1^j v_1^k N^{<ijk>} \alpha_4 / r^4 \end{aligned}$$

Non-singular 3PN self-integrals

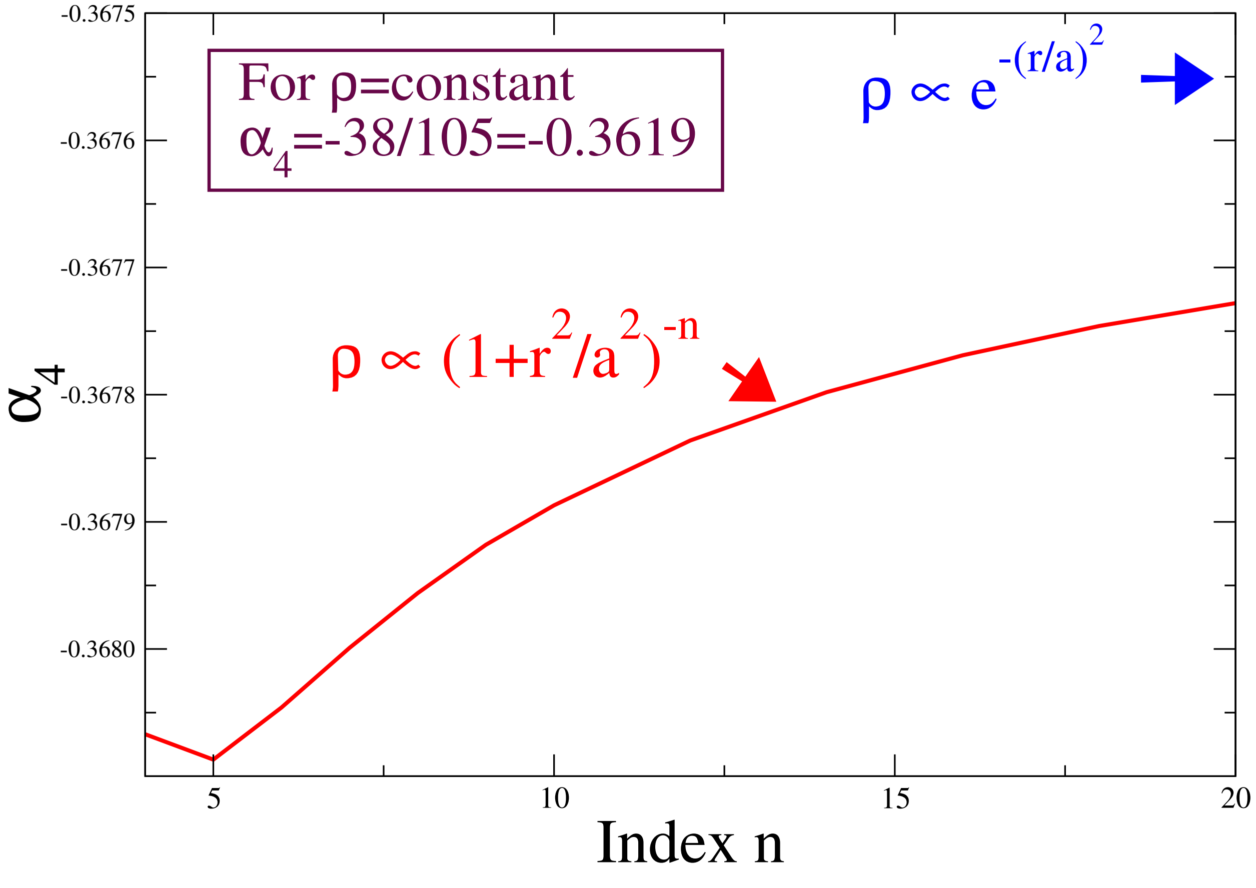
$$\text{Parameter} \;=\; \frac{1}{m^3}\int \rho(r) f(r) d^3x$$

Param	$f(r)$	Param	$f(r)$	Param	$f(r)$	Param	$f(r)$
α_1	UX	β_1	UY/r^2	γ_1	UZ/r^4	ζ_1	U^2r^2
α_2	$UX'r$	β_2	UY'/r	γ_2	UZ'/r^3	ζ_2	$UU'r^3$
α_3	$U'Xr$	β_3	$U'Y/r$	γ_3	$U'Z/r^3$		
α_4	$U'X'r^2$	β_4	$U'Y'$	γ_4	$U'Z'/r^2$	ζ_4	$U'U'r^4$
α_5	$U''Xr^2$	β_5	$U''Y$	γ_5	$U''Z/r^2$	ζ_5	$U''Ur^4$
α_6	$U''X'r^3$	β_6	$U''Y'r$	γ_6	$U''Z'/r$	ζ_6	$U''U'r^5$
η_1	r^2P_2	ω_1	r^2P_3	λ_1	F_2	ξ_1	F_3
η_2	r^3P_2'	ω_2	r^3P_3'	λ_2	rF_2'	ξ_2	rF_3'
η_3	r^2PT_2	ω_3	r^2PT_3	λ_3	FT_2	ξ_3	FT_3
η_4	r^3PT_2'	ω_4	r^3PT_3'	λ_4	rFT_2'	ξ_4	rFT_3'
μ_1	$\mathcal{G}$						
μ_2	$r\mathcal{G}'$						

Definitions of Functions (spherical symmetry)

$$\nabla^2 U = -4\pi\rho\,, \qquad \nabla^2 X = 2U\,, \quad \nabla^2 Y = 12X\,, \quad \nabla^2 Z = 30Y\,,$$
$$\nabla^2 P_2 = -(U')^2\,, \quad \nabla^2 PT_2 - 6PT_2/r^2 = -(U')^2\,,$$
$$\nabla^2 P_3 = 4\pi\rho U\,, \quad \nabla^2 PT_3 - 6PT_3/r^2 = -U(U'' - U'/r)\,,$$
$$\nabla^2 F_2 = 2P_2\,, \quad \nabla^2 FT_2 - 6FT_2/r^2 = 2PT_2\,,$$
$$\nabla^2 F_3 = 2P_3\,, \quad \nabla^2 FT_3 - 6FT_3/r^2 = 2PT_3\,,$$
$$\nabla^2 \mathcal{G} = -U^2\,.$$

The parameter α_4



EQUATIONS OF MOTION WITH SPIN & RADIATION REACTION

(CW & A. Gopakumar)

Baryonic Definitions

$$
\begin{aligned}
m_A x_A^i &\equiv \int_A \rho^* x^i d^3x \,, \quad \bar{x}^i = x^i - x_A^i \,, \quad \bar{v}^i = v^i - v_A^i \\
m_A &\equiv \int_A \rho^* d^3x \,, \quad S_A^i \equiv \epsilon^{ijk} \int_A \rho^* \bar{x}^j \bar{v}^k d^3x
\end{aligned}
$$

Equation of Motion

$$
\begin{aligned}
m_A a_A^i &= \int_A \rho^* (dv^i/dt) d^3x \\
&= \int_A \rho^* \left\{ \nabla^i U + a^i_{1PN} + \ldots + a^i_{2.5PN} + a^i_{3.5PN} \right\}
\end{aligned}
$$

$$SO/SS \qquad 0 \qquad SO/SS R.R.$$

Spin Evolution

$$
\begin{aligned}
dS_A^i/dt &= \epsilon^{ijk} \int_A \rho^* \bar{x}^j (dv^k/dt) d^3x \\
&= \epsilon^{ijk} \int_A \rho^* \bar{x}^j \left\{ \nabla^i U + a^i_{1PN} + \ldots + a^i_{2.5PN} + a^i_{3.5PN} \right\}
\end{aligned}
$$

$$SO/SS \qquad \text{“0”} \qquad SO/SS R.R.$$

Transform and Apply Results

- “Spin Supplementary Condition” arbitrariness
- Check $\dot{E}$ against KWW results with spin
- Check $\dot{J}$ against Kidder results with spin
- Study evolution of binary orbits with spin and RR.

DIRE: STATUS & FUTURE WORK

Done:

- Complete explication of DIRE and its foundations
  - *PRD* **62**, 124015 (2000) (gr-qc/0007087)
- Equations of motion for "point masses" through 2.5PN & 3.5PN order
  - *PRD* **65**, 104008 (2002) (gr-qc/0201001)

In Progress:

- Equations of motion for spinning bodies, with radiation reaction.
  - (Gopakumar, CMW)
- 3PN contributions to equations of motion
  - (H. Wang, J. Zeng, T. Mitchell, E. Gouillart, CMW)

Future Work:

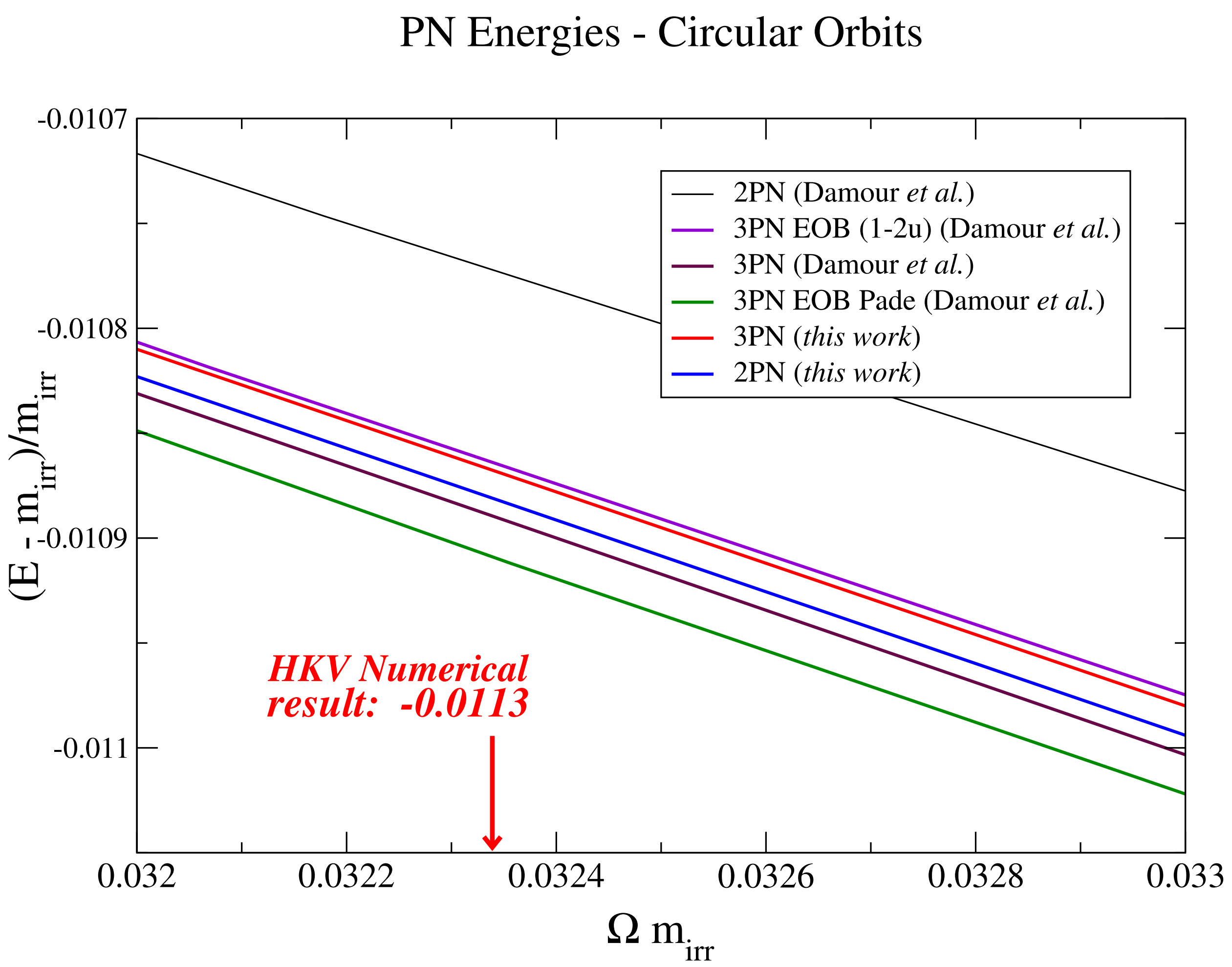
- 2.5PN and 3PN gravitational waveform
- Validity of the Strong Equivalence Principle at high PN order
- PN input for numerical relativity

THE END-GAME OF RADIATION REACTION

- Evolution leaves quasi-circular orbit
  - describable by PN approximation
- Numerical simulations start with quasi- equilibrium (QE) states
  - helical Killing vector $\partial/\partial t + \Omega \partial/\partial \phi$
  - stationary in rotating frame ($\dot{r} = 0$)
- How well do PN and QE agree?
  - Surprisingly well, but ...

But if QE state is actually an eccentric orbit at its apocenter ...

$\Rightarrow$ *Hint: Some NS-NS and BH-BH evolutions show oscillatory orbits*


PN Energies - Circular Orbits
2PN (Damour et al.)
3PN EOB (1-2u) (Damour et al.)
3PN (Damour et al.)
3PN EOB Pade (Damour et al.)
3PN (this work)
2PN (this work)
HKV Numerical result: -0.0113
(E - m_irr)/m_irr
Ω m_irr
-0.0107
-0.0108
-0.0109
-0.011
0.032
0.0322
0.0324
0.0326
0.0328
0.033

Newtonian orbit energy at apocenter

$$
\begin{aligned}
p/r &= 1 + e\cos(\phi - \omega) \\
&\to 1 - e \\
r^2\Omega &= (mp)^{1/2} \\
E/\mu &= (\dot{r}^2 + r^2\Omega^2)/2 - m/r \\
&\to r^2\Omega^2/2 - m/r
\end{aligned}
$$

Eliminate r and p:

$$
\boxed{
\begin{aligned}
E/\mu &= -\frac{1}{2}(1-e^2)\left[\frac{m\Omega_a}{(1-e)^2}\right]^{2/3} \\
J/\mu m &= \left[\frac{m\Omega_a}{(1-e)^2}\right]^{-1/3}
\end{aligned}
}
$$

Post-Newtonian orbit energy at apocenter

Define a "measurable" e and p

$$\boxed{\begin{aligned} e &\equiv \frac{\sqrt{\Omega_p} - \sqrt{\Omega_a}}{\sqrt{\Omega_p} + \sqrt{\Omega_a}}, \\ \zeta \equiv \frac{m}{p} &\equiv \left(\frac{\sqrt{m\Omega_p} + \sqrt{m\Omega_a}}{2} \right)^{4/3} \\ &= \left(\frac{m\Omega_a}{(1-e)^2} \right)^{2/3} \end{aligned}}$$

1 PN point-mass energy:

$$\begin{aligned} \frac{E}{\mu} &= -\frac{1}{2}\left(1 - e^2\right)\zeta \\ &\quad + \left[\frac{3}{8} + \frac{\eta}{24} + \left(-\frac{5}{12} + \frac{\eta}{12}\right) e^2 \right. \\ &\quad \left. + \left(\frac{1}{24} - \frac{\eta}{8}\right) e^4\right] \zeta^2 \end{aligned}$$

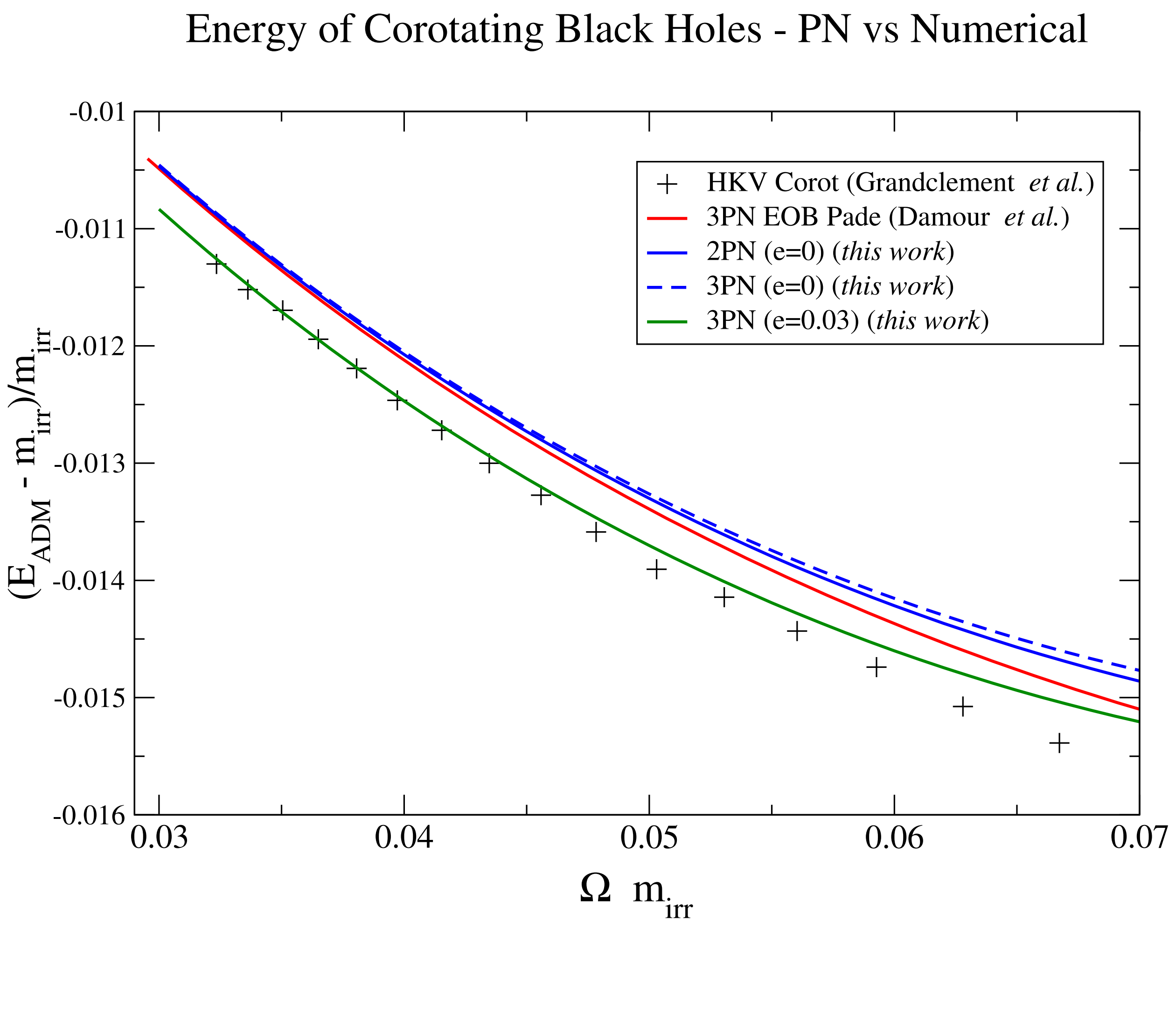
Augment with

- PN corrections through 3PN order
- Tidal interactions (not needed for BH systems)
- Rotational contributions
- Spin-orbit terms

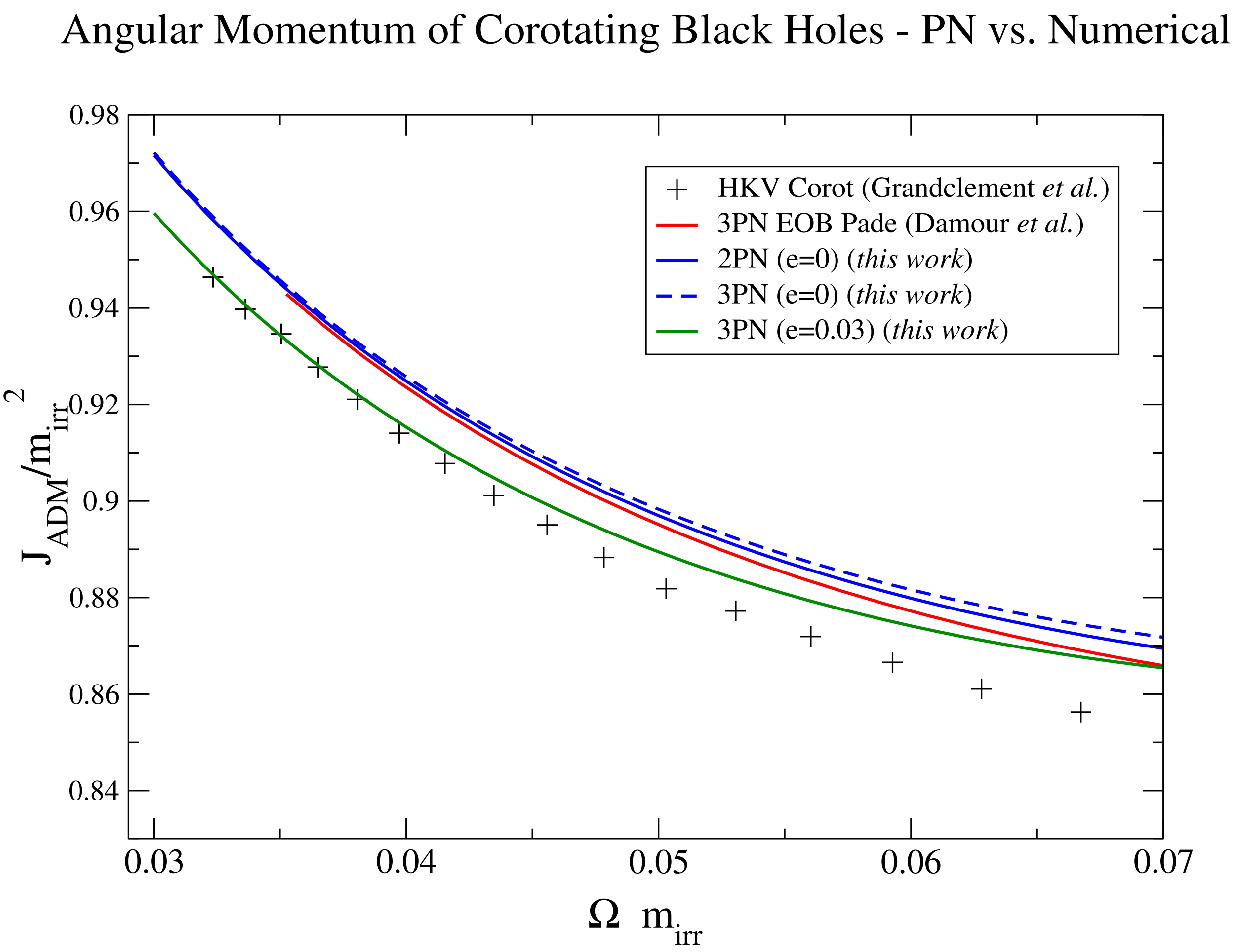
Compare with

- HKV initial data solutions of Grandclément, Gourgoulhon & Bonazzola
- PN circular orbit solutions of Blanchet, Damour, et al

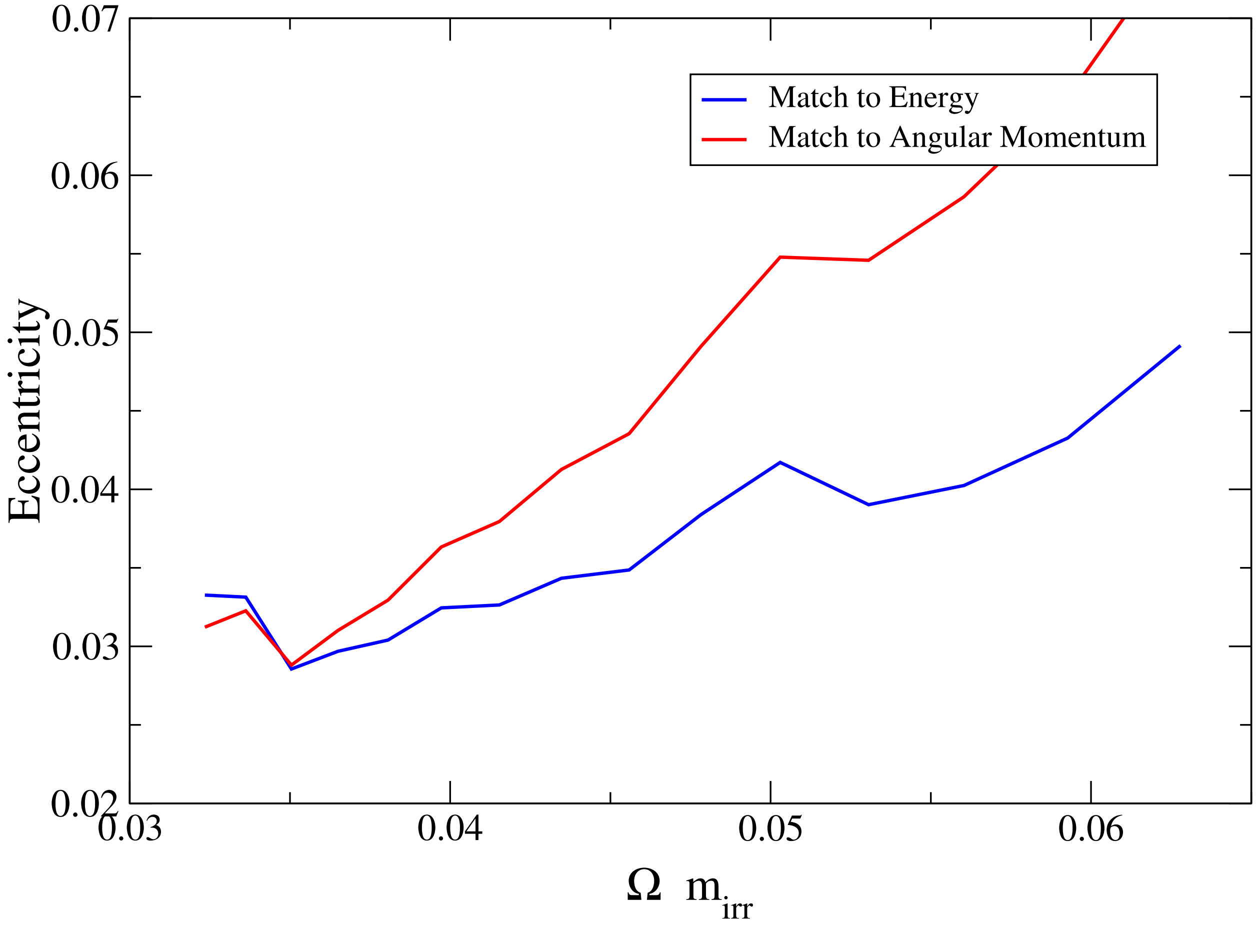
Mora & Will *PRD* **66**, 101501 (2002)



Energy of Corotating Black Holes - PN vs Numerical
HKV Corot (Grandclement et al.)
3PN EOB Pade (Damour et al.)
2PN (e=0) (this work)
3PN (e=0) (this work)
3PN (e=0.03) (this work)
-0.01
-0.011
-0.012
-0.013
-0.014
-0.015
-0.016
0.03
0.04
0.05
0.06
0.07
(E_ADM - m_irr)/m_irr
Ω m_irr



Angular Momentum of Corotating Black Holes - PN vs. Numerical
HKV Corot (Grandclement et al.)
3PN EOB Pade (Damour et al.)
2PN (e=0) (this work)
3PN (e=0) (this work)
3PN (e=0.03) (this work)
0.98
0.96
0.94
0.92
0.9
0.88
0.86
0.84
J_ADM/m_irr^2
0.03
0.04
0.05
0.06
0.07
Ω m_irr



Match to Energy
Match to Angular Momentum
Eccentricity
Ω m_irr
0.02
0.03
0.04
0.05
0.06
0.07
0.03
0.04
0.05
0.06

Questions, Issues, and Future Work

- Error bars needed on numerical simulations!
- Compare with other BH-BH quasi-equilibrium data
- Compare with NS-NS quasi-equilibrium data
  - tidal effects formally 5PN order, but important for NS
  - proportional to $(R/r)^5$
  - how good is a Newtonian treatment?
  - apsidal constant k_2 for neutron stars