FORMATION OF SINGULARITIES FOR YANG-MILLS EQUATIONS

- MOTIVATION (YANG-MILLS Eqs. AS A TOY-MODEL FOR EINSTEIN Eqs.)
- SETUP : YM EQUATIONS IN $d+1$ DIMENSIONS ; CAUCHY PROBLEM
- FORMATION OF SINGULARITIES IN $d=5$ (joint work with Z. TABOR)
- ANALYTIC UNDERSTANDING OF NUMERICS , CONJECTURES

FORMATION OF SINGULARITIES IN $d=4$, RATE OF BLOWUP (joint work with Z. TABOR, M. SIGAL, Yu. OVCHINNIKOV)

$$u_t = A(u) \qquad u(0) = u_0 \in \mathcal{D}$$

$$u_0 \xrightarrow{t} ?$$

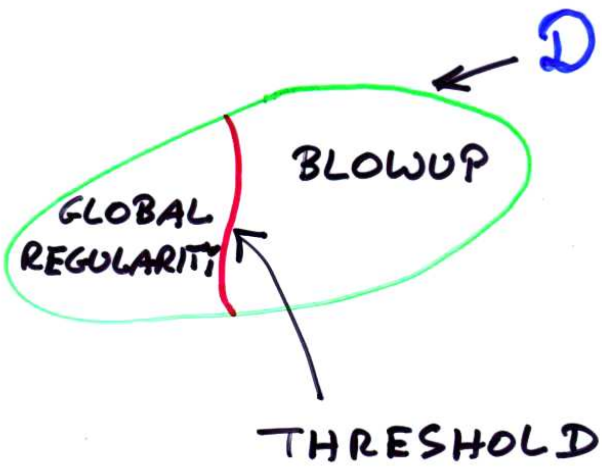
$$u_0$$

$\swarrow t$ $\qquad$ $t \searrow$

GLOBAL EXISTENCE $\qquad$ BREAKDOWN (BLOWUP)

$\downarrow$ $\qquad$ $\downarrow$

SCATTERING $\qquad$ CHARACTER OF BLOWUP



REGULARITY AND SCALING

$$u(x,t) \to u_\lambda(x,t) = \lambda^{-a} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^b}\right)$$

$$E(u_\lambda) = \lambda^\beta E(u)$$

$\beta < 0$ SUBCRITICAL (expect global regularity)

$\beta > 0$ SUPERCRITICAL (expect singularities)

$\beta = 0$ CRITICAL

$\beta_{YM} = d - 4$ $\beta_{Einstein} = d - 2$

YANG-MILLS IN $\underline{d = 5}$ IS THE RIGHT TOY-MODEL FOR THE EINSTEIN EQUATIONS IN THE PHYSICAL DIMENSION

YANG-MILLS IN d+1 DIMENSIONS

$A_\alpha : \mathbb{R}^{d+1} \to \mathfrak{g}$ Lie algebra of a Lie group G

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]$$

$$S = \frac{1}{e^2} \int \mathrm{Tr}\left(F_{\alpha\beta} F^{\alpha\beta}\right) d^d x \, dt$$

$$\partial_\alpha F^{\alpha\beta} + [A_\alpha, F^{\alpha\beta}] = 0$$

$$A_\alpha \to U^{-1} A_\alpha U + U^{-1} \partial_\alpha U \qquad U \in G$$

e — coupling constant

$$[e^2] = M^{-1} L^{d-4} \qquad (c=1)$$

$$[e^2]_{d=5} = M^{-1} L = [G]_{d=3}$$

Yang-Mills equations

$$\partial_\alpha F^{\alpha\beta} + [A_\alpha, F^{\alpha\beta}] = 0$$

are scale invariant:

If $x \to x/\lambda$, then $A_\alpha(x) \to \tilde{A}_\alpha(x) = \lambda^{-1} A_\alpha(x/\lambda)$

The energy

$$E(A) = \int_{R^d} Tr\left(F_{0i}^2 + F_{ij}^2\right)$$

scales as

$$E(\tilde{A}) = \lambda^{d-4} E(A)$$

Classification:

$d \leq 3$ subcritical (expect global regularity)

$d = 4$ critical (?)

$d \geq 5$ supercritical (expect singularities)

Known facts:

$d = 3$: Global existence for smooth initial data (Eardley and Moncrief, 1982); strengthened to finite-energy initial data (Klainerman and Machedon, 1995).

$d = 4$: Local well-posedness for small initial data in H^s, $s > 1$ (Klainerman and Tataru, 1999)

$d \geq 5$: Existence of self-similar solutions in $d = 5, 7, 9$ (Cazenave, Shatah, Tahvildar-Zadeh, 1998).

Assume $G = SO(d)$ so $g = so(d)$

A_α – $d \times d$ skew-symmetric matrices

Spherically-symmetric ansatz
$(i, j = 1, ..d,\ \alpha = 0, 1, ...d)$

$$A_\alpha^{ij}(x) = \left(\delta_\alpha^i x^j - \delta_\alpha^j x^i\right) \frac{1 - w(t,r)}{r^2}$$

Then the Cauchy problem reduces to

$$w_{tt} = \Delta_{(d-2)} w + \frac{d-2}{r^2} w(1 - w^2),$$

$$w(0,r) = f(r), \quad w_t(0,r) = g(r).$$

Boundary condition at the center

$$w(t,0) = 1 + O(r^2) \quad \text{as} \quad r \to 0.$$

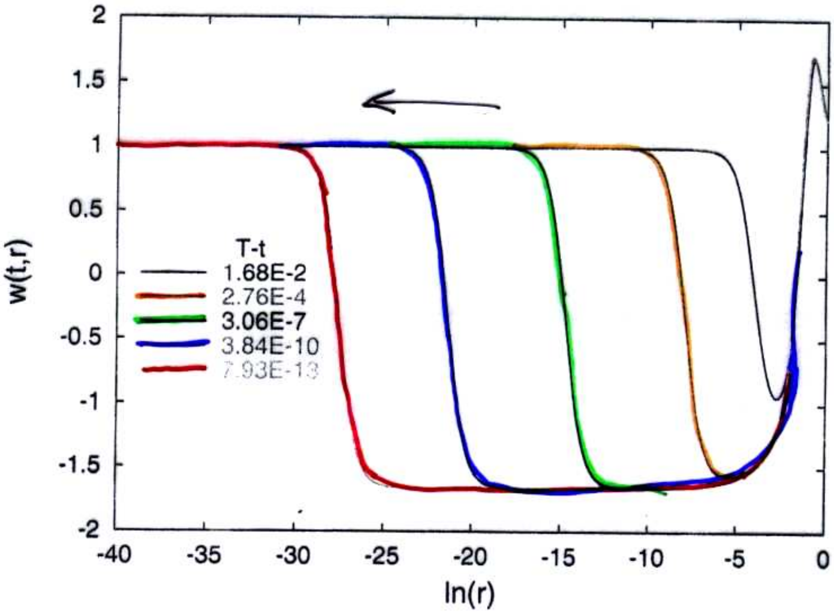
Typical initial data (time-symmetric or ingoing)

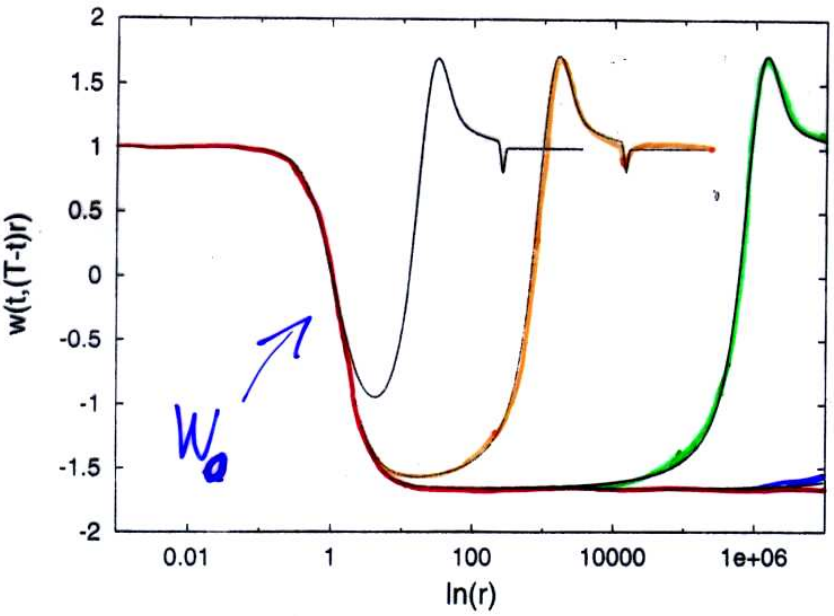
$$w(0,r) = 1 - A\, r^2 \exp\left[-\sigma(r-R)^2\right]$$

with adjustable amplitude A.

BLOWUP IN $d=5$

$|w_{rr}(t,0)| \to \infty$ as $t \to T$





Self-similar solutions in $d = 5$.

Assuming

$$w(t, r) = W(\eta), \qquad \eta = \frac{r}{T - t},$$

we get an ODE singular BVP

$$W'' + \frac{2}{\eta} W' + \frac{3}{\eta^2(1-\eta^2)} W(1 - W^2) = 0.$$

$$W(0) = \pm 1 \quad \text{and} \quad W(1) = 0.$$

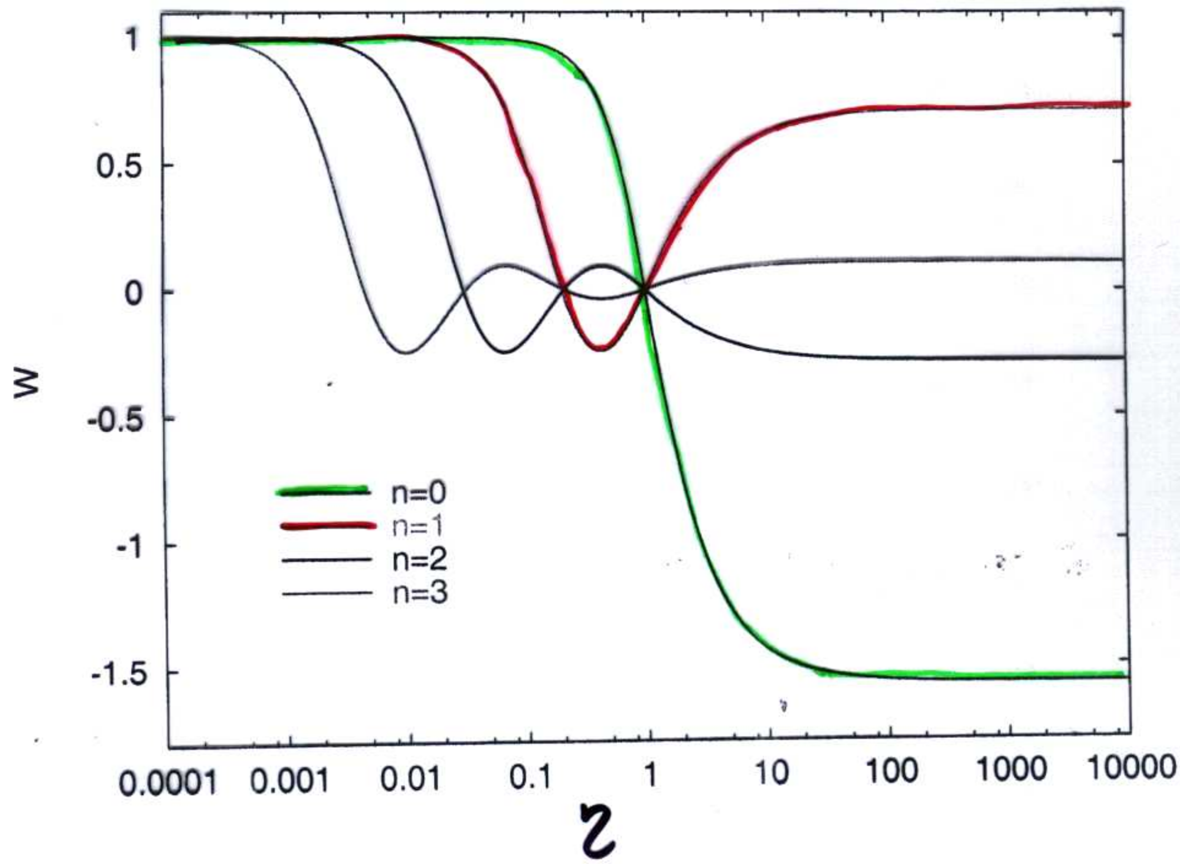
Theorem 1. *There exists a countable family of smooth self-similar solutions W_n satisfying the above BVP. The index $n = 0, 1, 2, \ldots$ denotes the number of zeros of $W_n(\eta)$ on the interval $0 < \eta < 1$. Moreover, the solution W_n has exactly n unstable modes.*

Proof of existence by a shooting technique.
Proof of instability using the zero mode.

$$w_{rr}(0, t) = \frac{1}{(T-t)^2} W''(0) \qquad \text{⑨}$$

$$\neq 0$$

Self-similar solutions in $d = 5$.



$$W_0(\eta) = \frac{1-\eta^2}{1+\frac{3}{5}\eta^2}$$

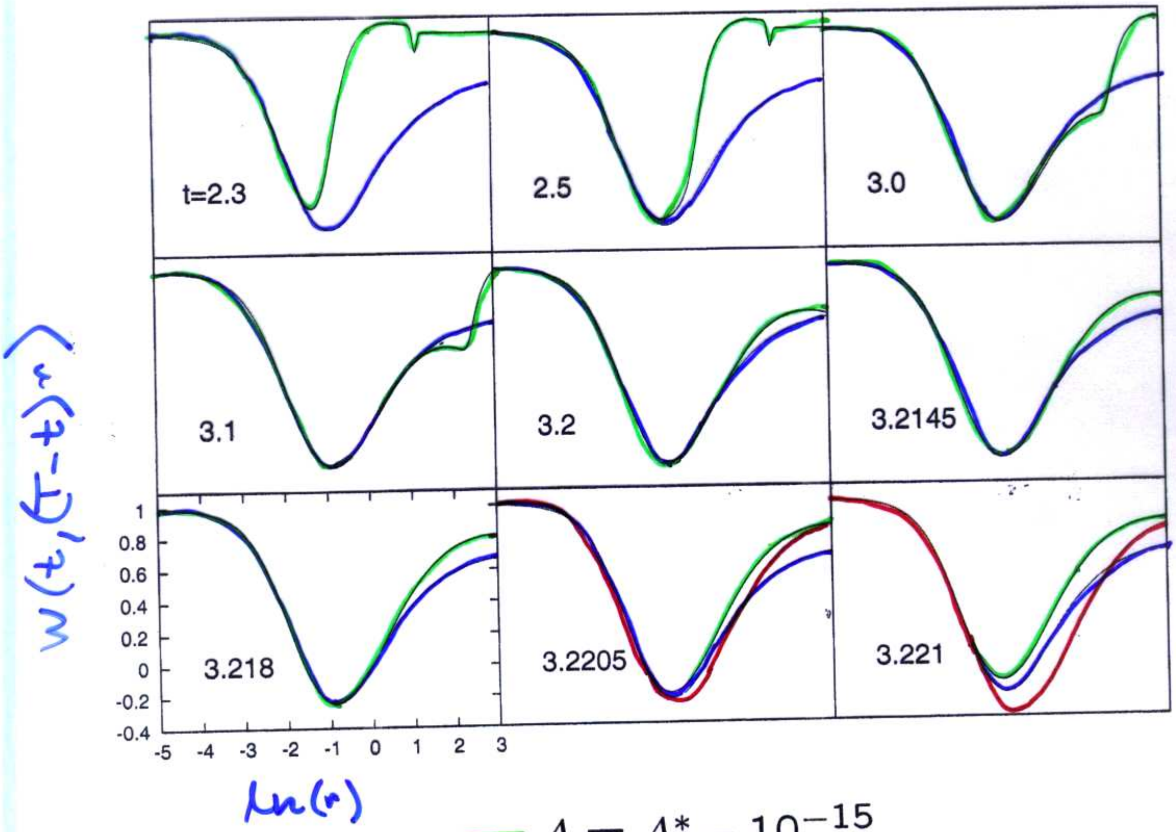
CONJECTURE (ON BLOWUP IN $d=5$)

SOLUTIONS WITH SUFFICIENTLY LARGE INITIAL DATA DO BLOW UP IN FINITE TIME IN THE SENSE THAT $w_{rr}(t,0)$ DIVERGES AS $t \nearrow T$ FOR SOME $T>0$. THE ASYMPTOTIC PROFILE OF BLOWUP IS GIVEN BY THE STABLE SELF-SIMILAR SOLUTION:

$$\lim_{t \nearrow T} w(t,(T-t)r) = W_0(r)$$

PROOF : ?

Critical behaviour



— $A = A^* - 10^{-15}$

— $A = A^* + 10^{-15}$

— W_1 - critical solution

INTERMEDIATE ASYMPTOTICS :

$$W(t,r) \approx W_1(\eta) + c(A)(T-t)^{-\alpha}\, v_1(\eta) + \text{DECAYING MODES}$$

v_1 – UNSTABLE EIGENMODE WITH POSITIVE EIGENVALUE α

$$c(A^*) = 0$$

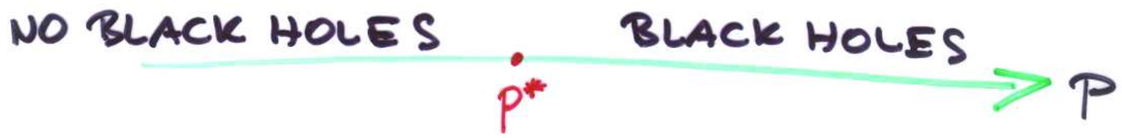
$$c(A)(T-t^*)^{-\alpha} \sim O(1) \Rightarrow (T-t^*) \sim |A-A^*|^{1/\alpha}$$

t^* → time of departure from W_1

SCALING LAW FOR MARGINALLY SUBCRITICAL DATA :

$$\max\{\text{energy density}(t,0)\} \sim (A^*-A)^{-4/\alpha}$$

CRITICAL PHENOMENA IN GRAVITATIONAL COLLAPSE



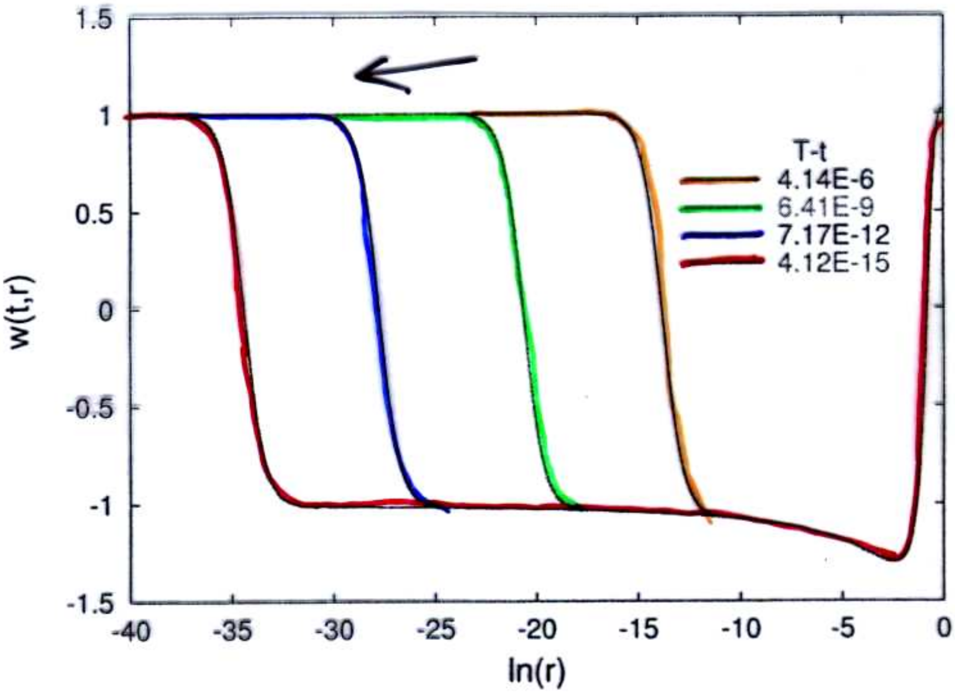
- UNIVERSALITY
- SELF-SIMILARITY (CRITICAL SOLUTION)
- SCALING

$$M_{BH} \sim (p - p^*)^\gamma$$

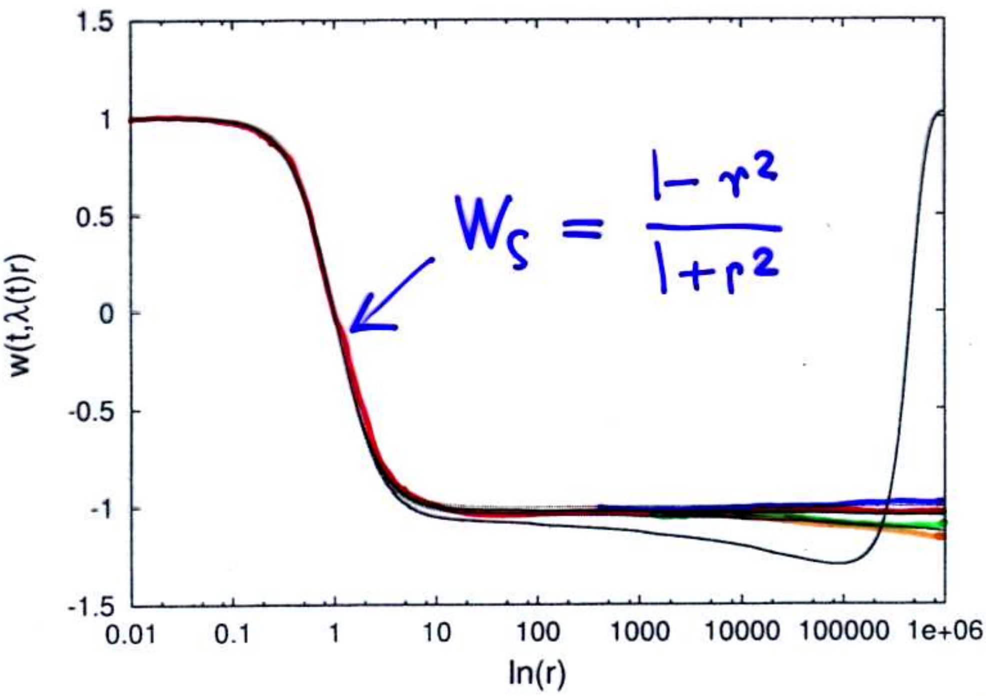
<u>LESSON</u>: CRITICAL PHENOMENA AT THE THRESHOLD FOR SINGULARITY FORMATION, ORIGINALLY FOUND FOR EINSTEIN'S EQUATIONS, SEEM TO BE ROBUST FEATURES OF SUPERCRITICAL PDEs.

ADVANTAGE: BETTER ANALYTIC INSIGHT – PROOF OF EXISTENCE OF A CRITICAL SOLUTION AND ITS SINGLE-MODE INSTABILITY.

Blowup in $d = 4$

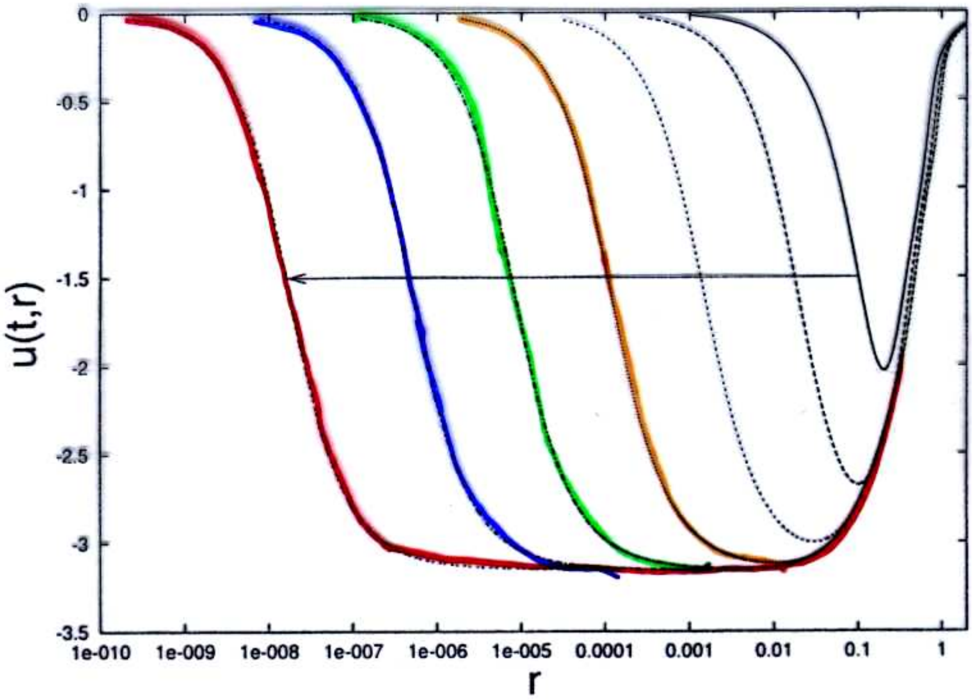


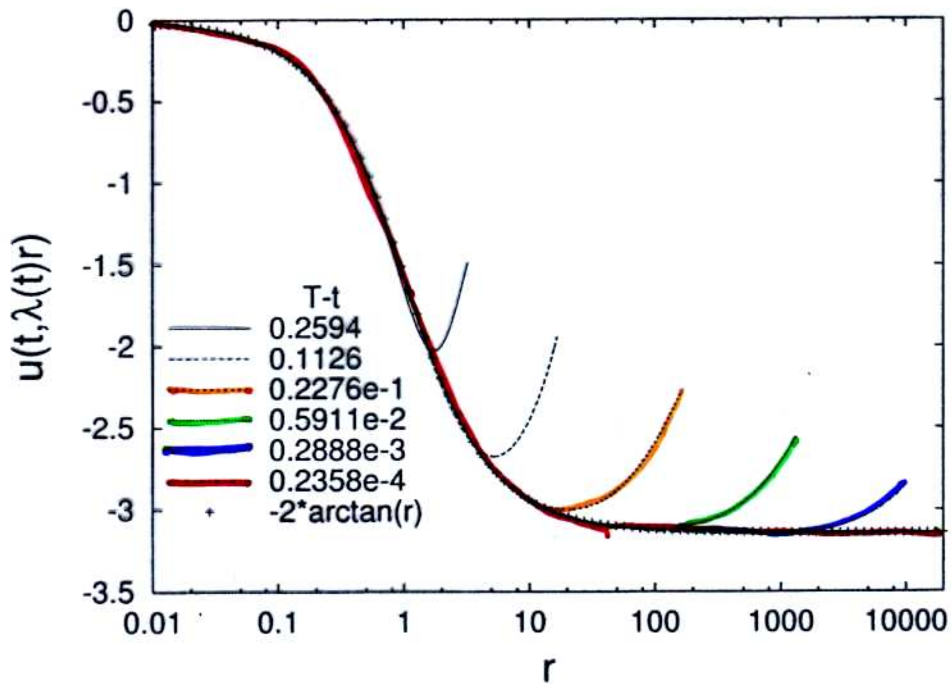
T-t
4.14E-6
6.41E-9
7.17E-12
4.12E-15
w(t,r)
ln(r)



$W_S = \frac{1-r^2}{1+r^2}$
w(t,λ(t)r)
ln(r)

Blowup for $2+1$ wave maps





CONJECTURE (ON BLOWUP IN $d=4$)

LARGE ENERGY SOLUTIONS DO BLOW UP IN FINITE TIME.
THE ASYMPTOTIC PROFILE OF BLOWUP IS GIVEN BY THE STATIC SOLUTION

$$\lim_{t \nearrow T} w(t, \lambda(t) r) = W_S(r) = \frac{1-r^2}{1+r^2}$$

WHERE $\lambda(t) \searrow 0$ AS $t \nearrow T$.

- WHAT IS THE RATE OF BLOWUP ?

$$\lambda(t) = ?$$

- WHAT IS THE THRESHOLD OF BLOWUP ?

GENERAL STRATEGY FOR SHOWING ASYMPTOTIC STABILITY OF SOLITONS :

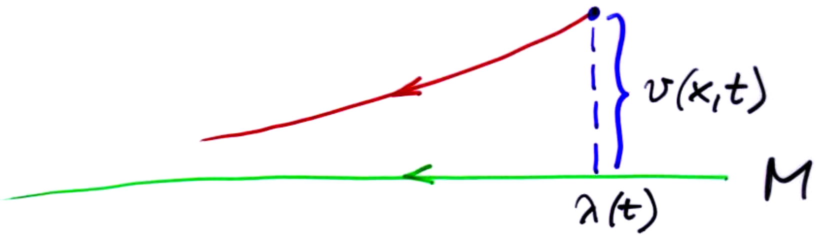
$$u_t = A(u) \quad , \quad A(u_s) = 0$$

STATIC SOLUTION

$$M = \{ u_s(x, \vec{\lambda}) \mid \vec{\lambda} \in \Lambda \}$$

MANIFOLD OF STATIC SOLUTIONS (MODULI SPACE)

PARAMETERS OF A GROUP OF SYMMETRY (COLLECTIVE COORDINATES)



$$u(x,t) = u_s(x, \vec{\lambda}(t)) + v(x,t)$$

"DISTANCE" FROM M

$$\boxed{v_t + \frac{\partial u_s}{\partial \vec{\lambda}} \frac{d\vec{\lambda}}{dt} = Lv + F(v, \vec{\lambda}) \qquad L = A'(u_s)}$$

ZERO MODES

INTRODUCE P = PROJECTION OPERATOR ON M, SO $Pv = 0$

THEN

$$\boxed{\begin{aligned} v_t &= Lv + P^{\perp} F(v, \vec{\lambda}) \\ \frac{d\vec{\lambda}}{dt} &= S(v, \vec{\lambda}) \end{aligned}}$$

$$W(t,\eta) = W_S(\eta) + v(t,\eta) \qquad \eta = r/\lambda(t).$$

$$\lambda^2\ddot{v} - 2\eta\dot{\lambda}\lambda\dot{v}' + Lv + N(v) = \lambda\ddot{\lambda}\eta W_S' - \dot{\lambda}^2(\eta^2 W_S'' + 2\eta W_S')$$

where

$$L = -\frac{\partial^2}{\partial\eta^2} - \frac{1}{\eta}\frac{\partial}{\partial\eta} - \frac{2(1-3W_S^2)}{\eta^2},$$

$$N(v) = \frac{6W_S}{\eta^2}v^2 + \frac{2}{\eta^2}v^3.$$

Orthogonal projection gives

$$Lv = -\dot{\lambda}^2(\eta^2 W_S'' + 2\eta W_S'),$$

so

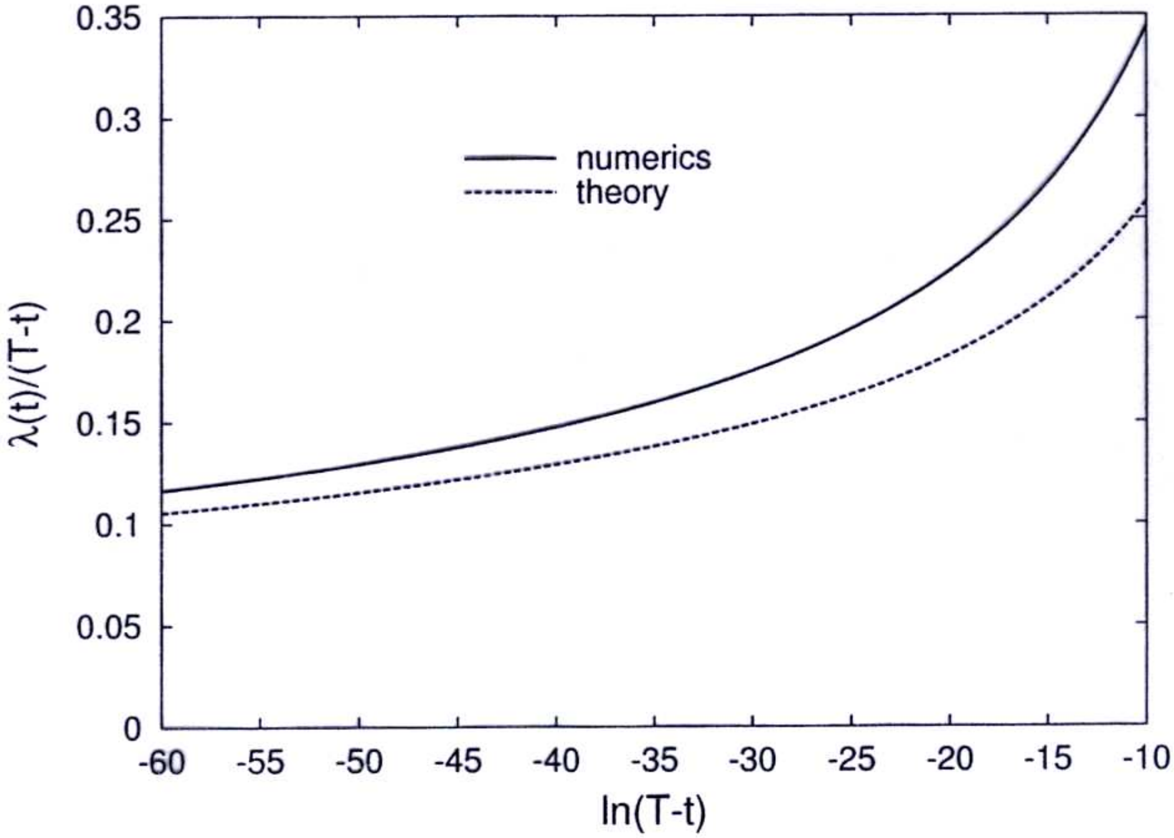
$$v \sim \dot{\lambda}^2.$$

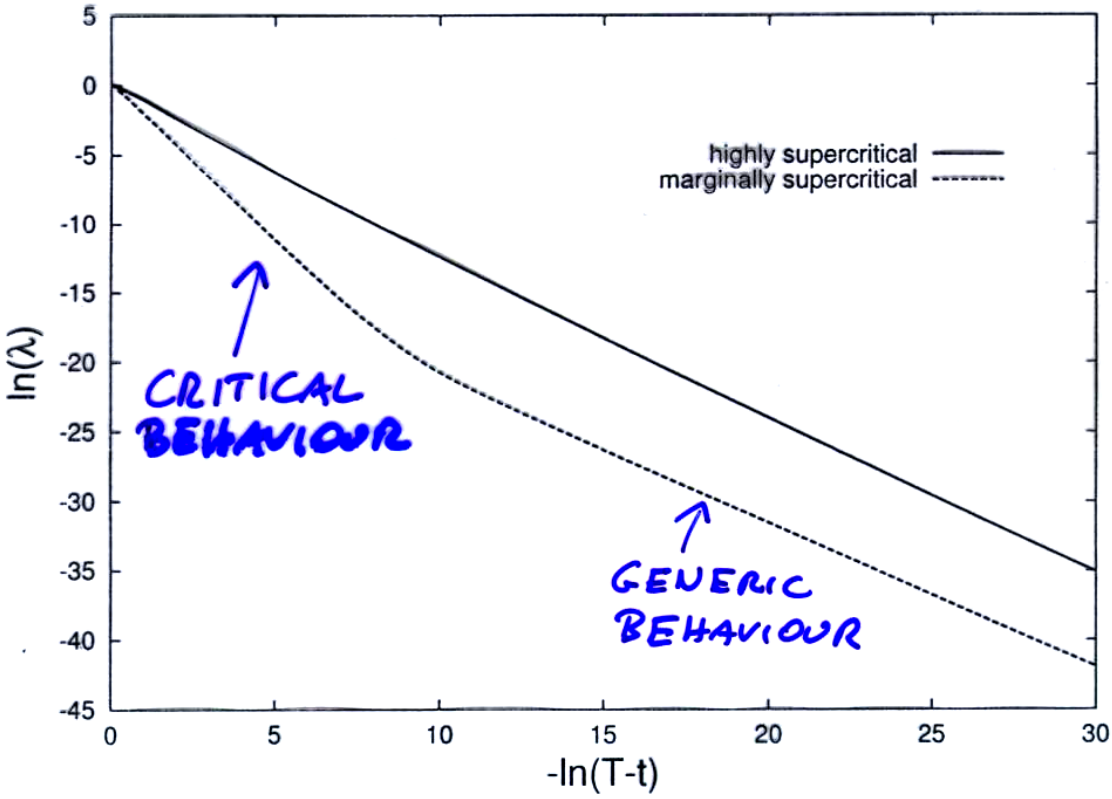
Projecting onto the zero mode we get

$$\boxed{\lambda\ddot{\lambda} = \frac{3}{4}\dot{\lambda}^4.}$$

The leading order solution for $t \to T$

$$\boxed{\lambda(t) \sim \sqrt{\frac{2}{3}}\frac{T-t}{\sqrt{-\ln(T-t)}}}$$


numerics
theory
λ(t)/(T-t)
ln(T-t)
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-60
-55
-50
-45
-40
-35
-30
-25
-20
-15
-10



THERE IS NO CRITICAL SOLUTION IS THE CRITICAL DIMENSION!